# *The Berger-Shaw theorem for a pair of commuting operators*

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# *hyponormal operators*

An operator $T$ on a Hilbert space $\mathscr{H}$ is said to be hyponormal if the commutator $[T^*,T] := T^*T - TT^*$ is positive.

The Berger-Shaw theorem says that if $T$ is a $m$-cyclic hyponormal operator, then the commutator $[T^*,T]$ is trace class and

$$\mathrm{tr}[T^*,T] \leq \frac{m}{\pi} A(\sigma(T))$$

There has been some attempt to show that if a commuting $n$-tuple of bounded linear operators $\boldsymbol{T}$ is hyponormal and cyclic, then the cross commutators must be trace class. The first of these is due to Athavale and the other is due to Douglas and Yan.

Douglas and Yan using techniques from commutative algebra reduce their proof to the case of a single operator.

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## strong and weak hyponormal

A commuting $n$ -tuple $\boldsymbol{T}$ of operators acting on a Hilbert space $\mathscr{H}$ is said to be $m$-cyclic if there exists a set of vectors $\zeta\{m\} := \{\zeta_1, \ldots, \zeta_m\}$ such that closed linear span of the vectors

$$\{p(T_1, \ldots, T_n)\zeta : \zeta \in \zeta\{m\}, p \in \mathbb{C}[z]\}.$$

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A commuting $n$ -tuple $\boldsymbol{T}$ of operators acting on a Hilbert space $\mathscr{H}$ is said to be strongly hyponormal if

$$[[\boldsymbol{T}^*, \boldsymbol{T}]] := \left(\left([\boldsymbol{T}_j^*, \boldsymbol{T}_i]\right)\right)_{i,j=1}^n : \oplus_n \mathscr{H} \longrightarrow \oplus_n \mathscr{H}$$

is positive, that is, for each $x \in \oplus_n \mathscr{H}$, $\langle [[T^*, T]]x, x \rangle \geq 0$, and it is said to be weakly hyponormal if for each vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, the sum $\sum_{i=1}^n \alpha_i T_i$ is a hyponormal operator on $\mathscr{H}$.

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## *question*

Question: If the $n$ -tuple $\boldsymbol{T}$ is strongly hyponormal and cyclic, then does it follow that the commutators $[T_j^*, T_i]$ , $1 \leq i,j \leq n$ is necessarily trace class?

It is easy to verify that the answer is "no", in general. Take for instance, the example of the Hardy space $H^2(\mathbb{D}^2)$ and the pair of operators to be the multiplication by the coordinate functions $(M_1, M_2)$ . Here the operators $M_j^* M_i - M_i M_j^* = 0$ , $j \neq i.$ However, the commutators $M_j^* M_j - M_j M_j^*$ are of infinite multiplicity and they are not even compact.

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# what we do

Instead of asking for the trace of the commutators to be finite, we only ask that the trace of a "certain" determinant (or, in the language of Helton and Howe, the generalized commutator) is finite.

One may argue that it is not asking for much. But then to arrive at this conclusion, we don't assume much either.

As in the Berger-Shaw theorem, we assume finite multiplicity but instead of either strong or weak hyponormality, we only assume that the determinant is positive. In many ways, it is a mild condition and this gives us the finiteness of the trace, what is more, we can even get an explicit bound.

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## *what is the determinant*

Let $B := ((B_{ij}))$ be an $n \times n$ block matrix with entries from $\mathscr{L}(\mathscr{H})$. The determinant of $B$ is the operator

$$\mathrm{Det}(B) := \sum_{\sigma,\tau} \mathrm{sgn}(\sigma) B_{\tau(1),\sigma(\tau(1))} B_{\tau(2),\sigma(\tau(2))}, \ldots, B_{\tau(n),\sigma(\tau(n))}.$$

The map $\mathrm{Det} : \mathscr{L}(\mathscr{H})^n \times \ldots \times \mathscr{L}(\mathscr{H})^n \mapsto \mathscr{L}(\mathscr{H})$ is clearly an alternating multi-linear map.

Let $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$ be a $n$-tuple of commuting operators. Let us say that the determinant of the $n$-tuple $\boldsymbol{T}$ is the operator $\mathrm{Det}([\![\boldsymbol{T}^*, \boldsymbol{T}]\!])$.

For operators of the form $[\![\boldsymbol{T}^*, \boldsymbol{T}]\!]$, Helton and Howe define the generalized commutator of $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$ : Let $A_1 = T_1^*, A_2 = T_1, \ldots, A_{2n-1} = T_n^*, A_{2n} = T_n$. The generalized commutator of the $n$-tuple $\boldsymbol{T}$ is the operator

$$GC(\boldsymbol{T}) := \sum_{\sigma} sgn(\sigma) A_{\sigma(1)} A_{\sigma(2)}, \ldots, A_{\sigma(2n)}.$$

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## *Det and GC are the same*

Thanks to Cherian Varughese, we see that $Det(\boldsymbol{T})$ and $GC(\boldsymbol{T})$ are equal, which is perhaps implicit in the paper of Helton and Howe.

Recall the example of the pair of multiplication operators on the Hardy space, $H^2(\mathbb{D}^2)$. In this case,

$$[[\boldsymbol{M}^*, \boldsymbol{M}]] = \begin{pmatrix} [(M_z \otimes I)^*, (M_z \otimes I)] & [(I \otimes M_z)^*, (M_z \otimes I)] \\ [(M_z \otimes I)^*, (I \otimes M_z)] & [(I \otimes M_z)^*, (I \otimes M_z)] \end{pmatrix}$$
$$= \begin{pmatrix} P \otimes I & 0 \\ 0 & I \otimes P \end{pmatrix} \geq 0.$$

It now follows that $Det([[\boldsymbol{M}^*, \boldsymbol{M}]]) = 2(P \otimes P)$.

Thus $Det([[\boldsymbol{M}^*, \boldsymbol{M}]])$ is positive and trace class.

indeed, $tr(Det[[\boldsymbol{M}^*, \boldsymbol{M}]]) = 2$.

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## a new class operators

Let $\boldsymbol{T} = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space $\mathscr{H}$ such that $\boldsymbol{T}$ is $m$-cyclic. Let $\zeta\{m\}$ be the minimal set of generating vectors for the pair $(T_1, T_2)$. Set

$$\mathscr{H}_N = \bigvee \left\{ T_1^{i_1} T_2^{i_2} v | v \in \zeta\{m\} \text{ and } 0 \leq i_1 + i_2 \leq N \right\}$$

and let $P_N$ be the projection onto $\mathscr{H}_N$.

Clearly, $P_N \uparrow_{SOT} I$.

A pair of commuting operators $\boldsymbol{T}$ is said to be in the class $BS_m(\mathscr{H})$ if $\boldsymbol{T}$ is $m$-cyclic and for every $N \in \mathbb{N}$, we have

$$\left\| P_N (T_1^* T_1 T_2^* - T_2^* T_1 T_1^*) P_N^{\perp} \right\| \leq \frac{1}{N+1} \left\| T_1 \right\|^2 \left\| T_2 \right\| \qquad (1)$$

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*Theorem*
*Let* $\boldsymbol{T} = (T_1, T_2)$ *be a pair of commuting operators on a Hilbert space* $\mathscr{H}$ *in the class* $BS_m(\mathscr{H})$. *If the determinant operator* $D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big)\,(= GC(\boldsymbol{T}))$ *is non negative definite then it is in trace-class and*

$$trace(D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big)) \leq \frac{2m}{\pi^2}\nu(\sigma(\boldsymbol{T})),$$

*where* $\nu$ *is the Lebesgue measure and* $\sigma(\boldsymbol{T})$ *is the Taylor-joint spectrum of the* $n$ *-tuple* $\boldsymbol{T}$.

## *the proof*

### *Lemma*

*Let* $\boldsymbol{T} = (T_1, T_2)$ *be a pair of commuting operators on a Hilbert space* $\mathscr{H}$ *such that* $\boldsymbol{T}$ *is* $m$ *-cyclic. Furthermore assume that* $\boldsymbol{T}$ *is in the class* $BS_m(\mathscr{H})$*. If the determinant operator* $D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big)$ *is positive then it is in trace-class and*

$$trace(D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big)) \leq 2m\|T_1\|^2\|T_2\|^2.$$

*Outline of the proof:*

$$\begin{aligned}&\mathrm{Det}\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big)\\ &\qquad = [T_1^*T_1T_2^*, T_2] - [T_1^*T_2T_2^*, T_1] + [T_2^*T_2T_1^*, T_1] - [T_2^*T_1T_1^*, T_2]\end{aligned}$$

For $j = 1, 2,$ note that $P_N T_j P_N^{\perp} = 0$ , $rank(P_N^{\perp} T_j P_N) \leq (N+1)m.$

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# *proof contd.*

For $i \neq j$, therefore we have

$$\begin{aligned}
P_N[T_i^* T_i T_j^*, T_j]P_N &= P_N(T_j T_i^* T_i T_j^* - T_i^* T_i T_j^* T_j)P_N \\
&= P_N T_j (P_N + P_N^{\perp}) T_i^* T_i T_j^*) P_N - P_N T_i^* T_i T_j^* (P_N + P_N^{\perp}) T_j P_N \\
&= [P_N T_i^* T_i T_j^* P_N, P_N T_j P_N] - P_N T_i^* T_i T_j^* P_N^{\perp} T_j P_N .
\end{aligned}$$

If $A, B$ are in trace-class, then $trace(AB) = trace(BA)$ and it follows that $trace([P_N T_i^* T_i T_j^* P_N, P_N T_j P_N]) = 0$. Hence

$$\begin{aligned}
\mathrm{trace}\big(P_N D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big) P_N\big) = \mathrm{trace}\big(P_N (T_2^* T_1 T_1^* - T_1^* T_1 T_2^*) P_N^{\perp} T_2 P_N\big) + \\
\mathrm{trace}\big(P_N (T_1^* T_2 T_2^* - T_2^* T_2 T_1^*) P_N^{\perp} T_1 P_N\big)
\end{aligned}$$

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Continuing, we have

$$\left|\mathrm{trace}(P_N D(\left[\left[\boldsymbol{T}^*,\boldsymbol{T}\right]\right])P_N)\right| \le \left\|P_N(T_1^*T_1T_2^* - T_2^*T_1T_1^*)P_N^{\perp}\right\|\left\|P_N^{\perp}T_2P_N\right\|_1 + \left\|P_N(T_2^*T_2T_1^* - T_1^*T_2T_2^*)P_N^{\perp}\right\|\left\|P_N^{\perp}T_1P_N\right\|_1,$$

where $\|\cdot\|_1$ is the trace norm.

Since $rank(P_N^{\perp}T_jP_N) \le (N+1)m, \left\|P_N^{\perp}T_jP_N\right\|_1 \le (N+1)m\|T_j\|$ and by definition $\left\|P_N(T_i^*T_iT_j^* - T_j^*T_iT_i^*)P_N^{\perp}\right\| \le \frac{1}{N+1}\|T_i\|^2\|T_j\|,$ we have

$$\left|\mathrm{trace}(P_N D(\left[\left[\boldsymbol{T}^*,\boldsymbol{T}\right]\right])P_N)\right| \le 2m\|T_1\|^2\|T_2\|^2.$$

But $D(\left[\left[\boldsymbol{T}^*,\boldsymbol{T}\right]\right])$ is non negative definite, therefore taking $P_N \uparrow I,$ we get

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$$\left|\mathrm{trace}(P_N D([[\boldsymbol{T}^*,\boldsymbol{T}]])P_N)\right| \le \left\|P_N(T_1^*T_1T_2^* - T_2^*T_1T_1^*)P_N^\perp\right\| \left\|P_N^\perp T_2 P_N\right\|_1 + \left\|P_N(T_2^*T_2T_1^* - T_1^*T_2T_2^*)P_N^\perp\right\| \left\|P_N^\perp T_1 P_N\right\|_1,$$

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# *Vitali covering*

For the second half of the theorem, we need two preparatory lemmas.

The first one says that if $\boldsymbol{T_i}$ is $m_i$ -multicyclic, $i = 1, 2,$ and $\sigma(T_1) \cap \sigma(T_2)$ is empty, then $T_1 \oplus T_2$ is $m$ -multi-cyclic, where $m = \max\{m_1, m_2\}$.

The other one is essentially the the Vitali covering lemma.

A Vitali covering of a finite measure space $(E, m)$ is a collection of closed balls $\mathscr{B}$ such that for each $x \in E$ and any $\varepsilon > 0,$ there is a $B \in \mathscr{B}$ with the property: $x \in B$ and $\nu(B) < \varepsilon.$

The Vitali covering Lemma says that if $(E, m)$ is a finite measure space and $\mathscr{B}$ is a "Vitali covering" of $E,$ then given any $\delta > 0,$ we can find finitely many disjoint balls $B_1, \ldots, B_N$ in $\mathscr{B}$ such that

$$\sum_{i=1}^{N} m(B_i) \geq m(E) - \delta.$$

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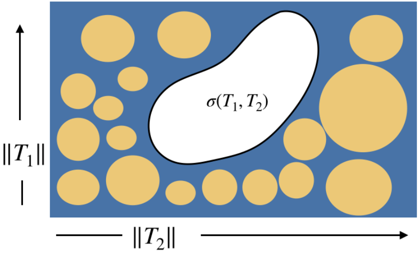
# *completing the proof – following that of Voiculescu*

Let $R_i = ||T_i||$ , $i = 1, 2,$ and put $\mathbb{D}_{12} = \mathbb{D}_{||T_1||} \times \mathbb{D}_{||T_2||}.$
Let $\varepsilon > 0,$ by Vitali covering lemma, there exist $B_1, \ldots, B_n$ pairwise disjoint balls in $\mathbb{D}_{12} \setminus \sigma(\boldsymbol{T})$ such that $\nu(\mathbb{D}_{12}) < \nu(\sigma(\boldsymbol{T})) + \sum_j \nu(B_j) + \varepsilon$




If $B_j = \bar{\mathbb{B}}(\boldsymbol{a}_j; r_j),$ where $\boldsymbol{a}_j \in \mathbb{C}^2$ the above inequality gives

$$\pi^2 ||T_1||^2 ||T_2||^2 - \frac{\pi^2}{2} \sum_j r_j^4 < \nu(\sigma(\boldsymbol{T})) + \varepsilon.$$

# *completing the proof*

Let $\boldsymbol{S}$ be the pair of shift operators on the Hardy space over ball $H^2(\mathbb{B}^2)$.

Define $L_j(\boldsymbol{Z}) = (\boldsymbol{a}_j + r_j\boldsymbol{Z})$.

Let $\boldsymbol{S}_j$ be the $m$-fold direct-sum copy of the operator $L_j(\boldsymbol{S})$ on the Hilbert space $H_j = \bigoplus_{j=1}^{m} H^2(B_j)$.

The pair $\boldsymbol{A} = \boldsymbol{T} \oplus \bigoplus_{i=1}^{n} \boldsymbol{S}_j$ is m-cyclic on the Hilbert space $\tilde{\mathscr{H}} = \mathscr{H} \oplus \bigoplus_{j=1}^{n} H_j$ since the spectrum of the summands are pairwise disjoint and each $\boldsymbol{S}_j$ is $m$-cyclic.

Clearly, $D\big([[\boldsymbol{A}^*, \boldsymbol{A}]]\big) = D\big([[\boldsymbol{T}^*, \boldsymbol{T}]]\big) \oplus \bigoplus D\big([[\boldsymbol{S}_j^*, \boldsymbol{S}_j]]\big)$ is non negative definite and $\|A_i\| = \|T_i\|$, $i = 1, 2$. For $i \neq j$, we have

$$\left\|\tilde{P}_N(A_i^*A_iA_j^* - A_j^*A_iA_i^*)\tilde{P}_N^{\perp}\right\| \leq \frac{1}{N+1}\left\|A_i\right\|^2\left\|A_j\right\|.$$

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## *completing the proof*

Thus all the hypothesis made for the pair $\boldsymbol{T}$ also holds good for the pair $\boldsymbol{A}$. Hence

$$\operatorname{trace}(D\big([[\boldsymbol{A}^*,\boldsymbol{A}]]\big)) \le 2m\|A_1\|^2\|A_2\|^2.$$

Now, it follows that

$$\begin{aligned}
\operatorname{trace}(D\big([[\boldsymbol{T}^*,\boldsymbol{T}]]\big)) &= \operatorname{trace}(D\big([[\boldsymbol{A}^*,\boldsymbol{A}]]\big)) - \sum_j \operatorname{trace}(D\big([[\boldsymbol{S}_j^*,\boldsymbol{S}_j]]\big)) \\
&\le 2m\|A_1\|^2\|A_2\|^2 - m\sum_j r_j^4 \\
&= \frac{2m}{\pi^2}\big(\pi^2\|T_1\|^2\|T_2\|^2 - \sum_j \frac{\pi^2}{2} r_j^4\big) \\
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# *More examples*

Let $\{e_{k,l}\}$ be an orthonormal basis $\ell^2(\mathbb{N}\times\mathbb{N})$ and $\boldsymbol{T}=(T_1,T_2)$ be a pair of joint weighted shifts:

$$T_1(e_{k,l}) = w^1_{k,l}e_{k+1,l}, \text{ where } w^1_{k,l} = \delta_k\sqrt{\tfrac{k-l+1}{k+2}}$$

$$T_2(e_{k,l}) = w^2_{k,l}e_{k+1,l+1}, \text{ where } w^2_{k,l} = \delta_k\sqrt{\tfrac{l+1}{k+2}}.$$

A simple computation gives: $D([[\boldsymbol{T}^*,\boldsymbol{T}]])e_{k,l} = \left(\frac{\delta_k^4}{k+2} - \frac{k\delta_{k-1}^4}{(k+1)^2}\right)e_{k,l}$.
It is then easy to verify that

$$trace\big(D([[\boldsymbol{T}^*,\boldsymbol{T}]])\big) \le r(T_\delta)^4,$$

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# More examples

Let $\{e_{k,l}\}$ be an orthonormal basis $\ell^2(\mathbb{N}\times\mathbb{N})$ and $\boldsymbol{T}=(T_1,T_2)$ be a pair of joint weighted shifts:

$T_1(e_{k,l})=w^1_{k,l}e_{k+1,l},$ where $w^1_{k,l}=\delta_k\sqrt{\frac{k-l+1}{k+2}}$

$T_2(e_{k,l})=w^2_{k,l}e_{k+1,l+1},$ where $w^2_{k,l}=\delta_k\sqrt{\frac{l+1}{k+2}}$.

A simple computation gives: $D([[\boldsymbol{T}^*,\boldsymbol{T}]])e_{k,l}=\left(\frac{\delta_k^4}{k+2}-\frac{k\delta_{k-1}^4}{(k+1)^2}\right)e_{k,l}$.
It is then easy to verify that

$$trace\big(D([[\boldsymbol{T}^*,\boldsymbol{T}]]\big)\leq r(T_\delta)^4,$$

where $r(T_\delta)$ denotes the spectral radius of the operator $T_\delta$ . This is in conformity with our inequality.

# Thank You!