Homogeneous Hermitian holomorphic vector bundles and the Cowen-Douglas class

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bounded symmetric domain

Let $\mathscr{D}$ be a bounded symmetric domain. The typical examples are the matrix unit ball $(\mathbb{C}^{n\times m})_1$ of size $n\times m,$ which includes the case of the Euclidean ball, i.e., $m=1.$

Let $G := \mathrm{Aut}(\mathscr{D})$ be the bi-holomorphic automorphism group of $\mathscr{D}.$

For the matrix unit ball, $G := SU(n,m),$ which consists of all linear automorphisms leaving the form $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ on $\mathbb{C}^{n+m}$ invariant.

Thus $g \in SU(n,m)$ is of the form $\begin{pmatrix} a & b \\ c & d. \end{pmatrix}$ The group $SU(n,m)$ acts on $(\mathbb{C}^{n\times m})_1$ via the map

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto (az+bz)(cz+dz)^{-1},\, z \in (\mathbb{C}^{n\times n})_1.$$

This action is transitive. Indeed $(\mathbb{C}^{n\times n})_1 \cong SU(n,n)/K,$ where K is the stabilizer of 0 in $(\mathbb{C}^{n\times n})_1.$

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imprimitivity

Let G be a locally compact second countable (lcsc) topological group and $\mathscr{D}$ be a lcsc G -space. Suppose that

$U : G \to \mathscr{U}(\mathscr{H})$ is a unitary representation of the group G on the on the Hilbert space $\mathscr{H}$ and that $\rho : C(\mathscr{D}) \to \mathscr{B}(\mathscr{H})$ is a $*$ - homomorphism of the C^* - algebra of continuous functions $C(\mathscr{D})$ on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators acting on the Hilbert space $\mathscr{H}$.

Then the pair (U, ρ) is said to be a representation of the G -space $\mathscr{D}$ if

$$\rho(\mathrm{g}\cdot\mathrm{f}) = \mathrm{U}(\mathrm{g})^{*}\rho(\mathrm{f})\mathrm{U}(\mathrm{g}), \mathrm{f} \in C(\mathscr{D}), \mathrm{g} \in \mathrm{G},$$

where $(\mathrm{g}\cdot\mathrm{f})(\mathrm{w}) = \mathrm{f}(\mathrm{g}^{-1}\cdot\mathrm{w}), \mathrm{w} \in \mathscr{D},$ which is a generalization of the imprimitivity relation due to Frobenius by Mackey.

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induction

As before, let K be the stabilizer group of 0 in $\mathrm{G},$ thus $\mathrm{G}/\mathrm{K} \cong \mathscr{D},$ where the identification is obtained via the map: $\mathrm{gK} \to \mathrm{g}0.$ The action of G on $\mathscr{D}$ is evidently transitive.

Given any unitary representation σ of $\mathrm{K},$ one may associate a representation $(\mathrm{U}^{\sigma}, \rho^{\sigma})$ of the G -space $\mathscr{D}.$ The correspondence

$$\sigma \to (\mathrm{U}^{\sigma}, \rho^{\sigma})$$

is an equivalence of categories.

The representation U^{σ} is the representation of G induced by the representation σ of the group $\mathrm{K}.$

For a semi-simple group $\mathrm{G},$ induction from the parabolic subgroups is the key to producing irreducible representations. Along with holomorphic induction, this method gives almost all the irreducible unitary representations of the semi-simple group $\mathrm{G}.$

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analytic Hilbert modules

Let $\mathscr{M} \subseteq \mathrm{Hol}(\mathscr{D})$ be a Hilbert space possessing a reproducing kernel, say, K.

Assume that $\mathscr{M}$ is a Hilbert module over the polynomial ring $\mathbb{C}[z]$. This means the map $(p, h) \to p \cdot h$, $p \in \mathbb{C}[z]$, $h \in \mathscr{M}$ defines a bounded operator for each fixed p.

In other words, $\rho(p) : h \mapsto p \cdot h$ defines a homomorphism $\rho : \mathbb{C}[z] \to \mathscr{L}(\mathscr{M})$.

One often assumes that the module map is continuous in both variables but we don't assume this. We make the standing assumption that $\mathscr{M}$ is an analytic Hilbert module over $\mathbb{C}[z]$.

Let $U : G \to \mathscr{U}(\mathscr{M})$ be a unitary representation.

What are the pairs (U, ρ) that satisfy the imprimitivity relation, namely,

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kernel function

Suppose that the kernel function K transforms according to the rule

$$J_g(z)K(g(z),g(w))J_g(w)^* = K(z,w),\ g \in G,\ z,w \in \mathscr{D},$$

for some holomorphic function $J_g : \mathscr{D} \to \mathbb{C}$.
Then the kernel K is said to be quasi-invariant, which is equivalent to saying that the map $U_g : f \to J_g\,(f \circ g^{-1}), g \in G,$ is unitary.
If we further assume that the $J_g : \mathscr{D} \to \mathbb{C}$ is a cocycle, then U is a homomorphism.

The pair (U, ρ) is a representation of the G-space $\mathscr{D}$ and conversely.
Therefore, our question becomes that of

> a characterization of all the quasi-invariant kernels defined on $\mathscr{D}$, or equivalently, finding all the holomorphic cocycles, which is also equivalent to finding all the holomorphic Hermitian homogeneous vector bundles over $\mathscr{D}$.

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example

Let B be the Bergman kernel of the domain $\mathscr{D}$ and pick $\lambda > 0$ such that B^λ is positive definite and the module map $h \xrightarrow{\rho(p)} p \cdot h, \; p \in \mathbb{C}[z],$ is bounded.

The Jacobian $j_g(z) : G \times \mathscr{D} \to \mathbb{C}, \; j_g(z) := \det(Dg^{-1}(z))$ defines a cocycle and it is to verify that the Bergman kernel is quasi-invariant relative to this co-cycle. Clearly, $B^\lambda, \lambda > 0,$ is quasi-invariant relative to j_g^λ.

Unless λ is a natural number, j_g^λ is not a cocycle. Never the less, it is a projective cocycle for all $\lambda > 0$. Consequently, the map $f \to j_g^\lambda f \circ g^{-1}$ on the Hilbert space $\mathbb{A}^{(\lambda)}(\mathscr{D})$ determined by the kernel B^λ is a projective unitary representation.

However, these projective representations can be realized as ordinary representations of the universal covering group $\tilde{G}$.

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complexification

Let $\mathfrak{g}$ be a simple non-compact real Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We have $\hat{z} \in \mathfrak{k}$ with $\mathfrak{z} = \mathbb{R}\hat{z} = \mathrm{center}(\mathfrak{k})$, and such that $\mathrm{ad}(\hat{z})$ defines a complex structure on $\mathfrak{p}$.

The complexification $\mathfrak{g}^{\mathbb{C}}$ is then the direct sum $\mathfrak{p}^{+} + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{-}$, where the components in the decomposition are the $i, 0, -i$ eigenspaces of $\mathrm{ad}(\hat{z})$, respectively.

We let $G^{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and we let $G, K^{\mathbb{C}}, K, P^{\pm}$ be the analytic subgroups corresponding to $\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}, \mathfrak{k}, \mathfrak{p}^{\pm}$. Then $\tilde{K}$, the universal cover of K, is also contained in $\tilde{K}^{\mathbb{C}}$, the universal cover of $K^{\mathbb{C}}$.

The two subalgebras $\mathfrak{p}^{\pm}$ are abelian and we let $P^{\pm}$ denote the corresponding analytic subgroups. The product $P^{+}K^{\mathbb{C}}P^{-}$ is open and dense in $G^{\mathbb{C}}$.

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The two subalgebras $\mathfrak{p}^{\pm}$ are abelian and we let $P^{\pm}$ denote the corresponding analytic subgroups. The product $P^+K^{\mathbb{C}}P^-$ is open and dense in $\mathbb{G}^{\mathbb{C}}$.

Harish-Chandra realization

Each $g \in P^+K^{\mathbb{C}}P^-$ admits a unique decomposition of the form $g = g^+g^0g^-$, where g^+, g^0 and g^- depend on g holomorphically. The map

$$\mathfrak{p}^+ \overset{\exp}{\to} P^+ \to G^{\mathbb{C}}/K^{\mathbb{C}}P^- : z \mapsto \exp z \mapsto (\exp z)K^{\mathbb{C}}P^-$$

imbeds the vector space $\mathfrak{p}^+$ holomorphically into $G^{\mathbb{C}}/K^{\mathbb{C}}P^-$.

Also, the natural map

$$G/K \to G^{\mathbb{C}}/K^{\mathbb{C}}P^- : gK \mapsto g(K^{\mathbb{C}}P^-)$$

is a holomorphic imbedding.

The image of this imbedding is contained in P^+, and applying $\exp^{-1}$ to it we obtain a bounded domain $\mathscr{D}$ in $\mathfrak{p}^+$. Writing $g \cdot z$ or gz for the action of $g \in \tilde{G}$ and $z \in \mathscr{D}$ by holomorphic automorphisms, we have $(g \exp z)^+ = \exp(g \cdot z)$.

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The image of this imbedding is contained in P^+, and applying $\exp^{-1}$ to it we obtain a bounded domain $\mathscr{D}$ in $\mathfrak{p}^+$. Writing $g \cdot z$ or gz for the action of $g \in \tilde{G}$ and $z \in \mathscr{D}$ by holomorphic automorphisms, we have $(g \exp z)^+ = \exp(g \cdot z)$.

Harish-Chandra realization

Each $g \in P^+K^{\mathbb{C}}P^-$ admits a unique decomposition of the form $g = g^+g^0g^-$, where g^+, g^0 and g^- depend on g holomorphically. The map

$$\mathfrak{p}^+ \stackrel{\exp}{\to} P^+ \to G^{\mathbb{C}}/K^{\mathbb{C}}P^- : z \mapsto \exp z \mapsto (\exp z)K^{\mathbb{C}}P^-$$

imbeds the vector space $\mathfrak{p}^+$ holomorphically into $G^{\mathbb{C}}/K^{\mathbb{C}}P^-$.

Also, the natural map

$$G/K \to G^{\mathbb{C}}/K^{\mathbb{C}}P^- : gK \mapsto g(K^{\mathbb{C}}P^-)$$

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notation

We will use the notations

$$\begin{aligned} k(g,z) &= g(\exp z)^0, \\ \exp Y(g,z) &= g(\exp z)^-. \end{aligned}$$

So the $P^+K^{\mathbb{C}}P^-$ decomposition of $g\exp z$ appears as

$$g\exp z = \exp(g\cdot z)k(g,z)\exp Y(g,z).$$

We also use the notation $b(g,z) = k(g,z)\exp Y(g,z)$. It is easy to see that

$$b(gh,z) = b(g,h\cdot z)b(h,z),\ g,h \in G,\ z \in \mathscr{D}.$$

The $\tilde{G}$ - homogeneous holomorphic vector bundles over $\mathscr{D}$ are obtained by holomorphic induction from representations (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ on finite dimensional vector spaces V. We write, as a standing notation, ρ^0, ρ^- for the restrictions of ρ to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{p}^-$, respectively.

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vector bundle

The representation space V is the orthogonal direct sum of its subspaces V_λ, $\lambda \in \mathbb{R}$, on which $\rho^0(\hat{z}) = i\lambda$. It is easy to see that $\rho^-(Y)V_\lambda \subset V_{\lambda-1}$ for $Y \in \mathfrak{p}^-$.

We assume each subspace V_λ is irreducible under $\mathfrak{k}^{\mathbb{C}}$. We call (ρ, V) and the induced bundle, indecomposable if it is not the orthogonal sum of sub-representations, respectively, sub-bundles. We restrict ourselves to describing these.

Proposition

Every indecomposable holomorphic homogeneous Hermitian vector bundle E can be written as a tensor product $L_{\chi_{\lambda_0}} \otimes E'$, where $L_{\chi_{\lambda_0}}$ is the line bundle induced by a character χ_{λ_0} and E' is the lift to $\tilde{G}$ of a G - homogeneous holomorphic Hermitian vector bundle.

This comes from the restriction to G and $\mathscr{D}$ of a $G^{\mathbb{C}}$ - homogeneous vector bundle induced in the holomorphic category by a representation of $K^{\mathbb{C}}P^-$.

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Given a representation (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$, the holomorphically induced bundle has a canonical trivialization such that the sections are the elements of $\mathrm{Hol}(\mathscr{D}, V)$, and $\tilde{G}$ acts via the multiplier

$$\rho(\tilde{b}(g,z)) = \rho^0(\tilde{k}(g,z))\rho^-(\exp Y(g,z)).$$

The representation (ρ, V) is a direct sum of subspaces $V_J := V_{\lambda - j}$, carrying an irreducible representation ρ^0_j of $\mathfrak{k}^{\mathbb{C}}\,(0 \leq j \leq m)$, also, we have non-zero $\mathfrak{k}^{\mathbb{C}}$- equivariant maps $\rho^-_j : \mathfrak{p}^- \to \mathrm{Hom}(V_{j-1}, V_j)$.

The space of such maps is 1 -dimensional: This is an equivalent restatement of the known fact that $\mathfrak{p}^- \otimes V_{j-1}$ as a representation of $\mathfrak{k}^{\mathbb{C}}$ is multiplicity free.

trivialization

Given a representation (ρ, V) of $\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{-}$, the holomorphically induced bundle has a canonical trivialization such that the sections are the elements of $\mathrm{Hol}(\mathscr{D}, V)$, and $\tilde{G}$ acts via the multiplier

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the non-trivial data

Let P_j be the orthogonal projection from $\mathfrak{p}^- \otimes V_{j-1}$ to V_j. We define for $Y \in \mathfrak{p}^-, v \in V_{j-1}$,

$$\tilde{\rho}_j(Y)v = P_j(Y \otimes v).$$

Then $\tilde{\rho}_j$ has the $\mathfrak{k}^{\mathbb{C}}$ -equivariant property, and it follows that $\rho_j^- = y_j \tilde{\rho}_j$ with some $y_j \neq 0$.

We write $y = (y_1, \ldots, y_m)$ and denote by E^y the induced vector bundle. We observe here that the vector bundle E^y is uniquely determined by $\rho_0^0, P_1, \ldots, P_m$ and y.

This data cannot be arbitrarily chosen: The $\tilde{\rho}_j \, (1 \leq j \leq m)$ together must give a representation of the abelian Lie algebra $\mathfrak{p}^-$.

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Theorem

There exists positive constants c_{jk}, the operator $\Gamma : \mathrm{Hol}(\mathscr{D}, V) \to \mathrm{Hol}(\mathscr{D}, V)$ given by

$$(\Gamma f_j)_\ell = \begin{cases} c_{\ell j} y_\ell \cdots y_{j+1} (P_\ell \imath D) \cdots (P_{j+1} \imath D) f_j & \text{if } \ell > j, \\ f_j & \text{if } \ell = j, \\ 0 & \text{if } \ell < j \end{cases}$$

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quasi-invariant reproducing kernels

Theorem

The sections of E^y have a $\tilde{G}$-invariant regular inner-product if and only if the same is true for E^0. In this case, the map Γ is a unitary isomorphism of $\mathscr{H}^0$ onto the Hilbert space $\mathscr{H}^y$ of sections of E^y. The space $\mathscr{H}^y$ (as well as $\mathscr{H}^0$) has a reproducing kernel.

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Cowen-Douglas operators

For a bounded symmetric $\mathscr{D}$, we call a n-tuple T in $\mathrm{B}'_{\mathrm{k}}(\mathscr{D})$ and its corresponding bundle E basic if E is induced by an irreducible ρ.

When $\mathscr{D}$ is the unit ball in $\mathbb{C}^{\mathrm{n}}$, E is basic if and only if it is induced by some $\chi_\lambda \otimes \sigma$ with $\lambda < \sigma_\lambda$.

Theorem

If $\mathscr{D}$ is the unit ball in $\mathbb{C}^{\mathrm{n}}$, all homogenous n - tuples in $\mathrm{B}'_{\mathrm{k}}(\mathscr{D})$ are similar to direct sums of basic homogeneous n - tuples.

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Thank You!