Mackey Imprimitivity and commuting tuples of homogeneous normal operators

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G- space

- Let G be a locally compact second countable group. A locally compact Hausdorff topological space $\mathbb{S}$ is said to be a G- space if there is a map $\alpha : G \times \mathbb{S} \to \mathbb{S}$, such that for a fixed $g \in G$, the map $s \to \alpha(g, s)$ is a bijective continuous map of $\mathbb{S}$ and $g \to \alpha_g$, $\alpha_g(s) := \alpha(g, s), g \in G,$ is a homomorphism.

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- Let $H \subseteq G$ be a closed subgroup and let $\mathbb{S} := G/H$ be the space of cosets: $\{gH \mid g \in G\}$. Equipped with the action of G by left multiplication: $g'(gH) := (g'g)H, g', g \in G$, the coset space $\mathbb{S}$ is a transitive G- space.

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- On the other hand, any transitive G- space must be of this form. In this case, we identify $\mathbb{S}$ with G/H and define $(g \cdot f)(s) = f(g^{-1} \cdot s)$ for any function defined on $\mathbb{S}$.

imprimitivity

- Let $C_0(\mathbb{S})$ be the algebra of continuous functions vanishing at ∞ on $\mathbb{S}$ and $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$.

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- Suppose that ρ is a $*$- homomorphism from $C_0(\mathbb{S})$ into $\mathcal{L}(\mathcal{H})$ and U is a unitary representation of the group G on the same Hilbert space $\mathcal{H}$. Then the imprimitivity is the relationship

$$U(g)\rho(f)U(g)^* = \rho(g \cdot f),\ g \in G,\ f \in C_0(\mathbb{S}),$$

where $g \cdot f$ is the function: $(g \cdot f)(s) = f(g^{-1} \cdot s), s \in \mathbb{S}$.

homogeneous normal

- A commuting d- tuple $\boldsymbol{N} = (N_1, \ldots, N_d)$ of normal operators acting on a complex separable Hilbert space $\mathcal{H}$ is said to be homogeneous with respect to a group G if the joint spectrum $\sigma_{\boldsymbol{N}} \subset \mathbb{C}^d$ is a G- space and there is a unitary representation U of G on $\mathcal{H}$ such that

$$\begin{aligned} U(g)^* \boldsymbol{N} U(g) &:= (U(g)^* N_1 U(g), \ldots, U(g)^* N_d U(g)) \\ &= (g_1(\boldsymbol{N}), \ldots, g_d(\boldsymbol{N})) := g(\boldsymbol{N}), \end{aligned}$$

  where $g_i, 1 \leqslant i \leqslant d$, are the coordinate functions of the action of G on $\sigma_{\boldsymbol{N}}$, namely,

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- The d- tuple $(\lambda_1, \cdots, \lambda_d)$ of complex numbers is said to be in $\sigma_{\boldsymbol{N}}$ if there exists a sequence x_n of unit vectors in $\mathcal{H}$ such that $(N_j - \lambda_j)\, x_n \to 0, 1 \leqslant j \leqslant d$.

imprimitivity theorem of Mackey

- The imprimitivity theorem of Mackey has two parts. First, any transitive imprimitivity $(\mathbb{S}, U, \rho)$ is equivalent to a canonical imprimitivity, where $\rho(f)$ for $f \in C_0(\mathbb{S})$ is defined to be the operator M_f of multiplication by f on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$ and U is a multiplier representation on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$, that is,
$$\big(U(g)h\big)(s) = c(g, s)(g \cdot h)(s), \;\; h \in L^2(\mathbb{S}, \mu, \mathcal{H}_n), \;\; g \in G,$$
where $c : G \times \mathbb{S} \to \mathcal{U}(\mathcal{H}_n)$ is a Borel map taking values in the group of unitary operators $\mathcal{U}(\mathcal{H}_n)$ of the Hilbert space $\mathcal{H}_n$ of dimension n.

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- For U to be a homomorphism, the function c must be a cocycle.
- The second part of the imprimitivity theorem asserts that such a multiplier representation is induced from a unitary representation of the subgroup H acting on the Hilbert space $\mathcal{H}_n$.
- It is evident that the d- tuple of multiplication by coordinate functions $(M_1, \dots, M_d)$ acting on $L^2(\mathbb{S}, \mu, \mathcal{H}_n)$ is homogeneous.

imprimitivity and homogeneous subnormal

We prove that any d- tuple $\boldsymbol{N}$ of commuting homogeneous normal operators such that $\sigma(\boldsymbol{N})$ is a G- space is determined by a direct sum of several transitive imprimitivities that may be taken to be of the canonical form without loss of generality and conversely.

example

- As an example, let G be the bi-holomorphic automorphism group of the unit disc. Thus $g \in G$ is a holomorphic map of the unit disc $\mathbb{D}$ of the form

$$g(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},\ 0 \leq \theta < 2\pi,\ a \in \mathbb{D}.$$

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- In this example, the action of the group on the closed unit disc $\bar{\mathbb{D}}$ is not transitive. However, it acts transitively on the open unit disc $\mathbb{D}$ and the boundary, the unit circle $\mathbb{T}$ separately. Thus we set out to study some class of imprimitivities that do not come from a transitive action. In this case, we expect an imprimitivity based on $\bar{\mathbb{D}}$ to be the direct sum of irreducible imprimitivities based on $\mathbb{D}$ and $\mathbb{T}$, namely, the direct sum of the multiplication by the coordinate function on

$$\oplus_m L^2(\mathbb{D}, dA) \oplus \oplus_{m'} L^2(\mathbb{T}, d\theta).$$

- This turns out to be true in much greater generality and this is what we discuss. With mild hypothesis, an imprimitivity is the direct sum of transitive imprimitivities. I will finish by discussing several examples and list all the homogeneous d- tuples of normal operators modulo unitary equivalence.

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- Thus imprimitivities, in general, are direct sums of transitive imprimtivities!

A spectral measure defined on $\mathbb{S}$ is a projection valued map $P : \mathcal{B} \to \mathcal{P}(\mathcal{H})$ such that

$$P(\mathbb{S}) = I \text{ and } P(\cup E_k) = \sum_{k=1}^{\infty} P(E_k)$$

for all disjoint collection of sets E_k, $k = 1, 2, \ldots,$ in $\mathcal{B}$, where the convergence is in the strong operator topology.

If P is a spectral measure for $(\mathbb{S}, \mathcal{B})$ and $x, y \in \mathcal{H}$, then

$$P_{x,y}(E) \equiv \langle P(S)x, y\rangle, \; x, y \in \mathcal{H}; \; S \in \mathcal{B},$$

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spectral theorem

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Theorem

Suppose that $\mathbb{S}$ is a locally compact Hausdorff space, and ρ is a nondegenerate $*$- representation of $C_0(\mathbb{S})$ on $\mathcal{H}$. Then there is a unique regular projection-valued measure P on $\mathbb{S}$ such that $\rho(f) = \int f dP$ for all $f \in C_0(\mathbb{S})$.

quasi-invariant measures

- Let $(\mathbb{S}, \mathcal{B})$ be the Borel measurable space, and note that each $g \in G$ defines a continuous map on $\mathbb{S}$ by our assumption. Given a σ- finite measure μ on $\mathbb{S}$, define the push-forward $g_*\mu$ of the measure μ by the requirement

$$(g_*\mu)(A) := \mu\big(g \cdot A\big),\ \ g \cdot A := \{g^{-1} \cdot s \mid s \in A\}, A \in \mathcal{B}.$$

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- The measure μ on G is said to be invariant if $g_*\mu = \mu$ and quasi-invariant if $g_*\mu$ is equivalent (mutually absolutely continuous) to μ for all $g \in G$.

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- There is a quasi-invariant measure uniquely determined modulo mutual absolute equivalence on $\mathbb{S}$.

multiplier representations

- Let $m : G \times \mathbb{S} \to \mathcal{U}(V)$ be a Borel function, where $\mathcal{U}(V)$ is the space of unitary operators on a complex separable Hilbert space V. Define

$$T_g f(x) = \left(\frac{d(g_* \mu)}{d\mu}(x) \right)^{\frac{1}{2}} m(g, x) f(g^{-1} \cdot x),$$

where f comes from $L^2(\mathbb{S}, \mu, V)$. We assume that T_g defines a unitary representation of G.

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- It is easily verified that $g \to T_g$ is a homomorphism if and only if the multiplier m satisfies the cocycle identity:

$$m(g_1 g_2, x) = m(g_1, x) m(g_2, g_1^{-1} \cdot x), \quad g_1, g_2 \in G, x \in \mathbb{S}.$$

Fix $x_0 \in \mathbb{S}$ and let H be the stabilizer group of x_0. Next, set $\sigma(g) = m(g, x_0)$, Notice that,

$$\sigma(hg) = \sigma(h)\sigma(g), \quad h \in H, g \in G.$$

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$$\sigma(hg) = \sigma(h)\sigma(g), \quad h \in H, g \in G.$$
- In particular, σ restricted to H is a homomorphism of H into $\mathcal{U}(V)$ the group of unitary operators on V and hence a unitary representation of H as m is Borel.

Let $\mathbb{S} = G/H$ be a homogeneous G- space and μ be a quasi-invariant measure on $\mathbb{S}$ (there is always one such uniquely determined modulo mutual absolute equivalence). Assume that $(\mathbb{S}, U, \rho)$ is an imprimitivity acting on some separable complex Hilbert space $\mathcal{H}$. Then there is a Hilbert space V such that the Hilbert space $\mathcal{H}$ is isometric to $\mathcal{H} = L^2(\mathbb{S}, \mu, V)$, where

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(iii) the representation U is of the form

$$(U(g)f)(s) = \sqrt{\frac{d\mu(g \cdot s)}{d\mu(s)}}\sigma(h)f(g \cdot s).$$

Here $h \in H$ is determined from the relation $g\,p(g^{-1} \cdot s) = p(s)h$, $s \in \mathbb{S}$, where $p : G/H \to G$ is a Borel cross-section.

the correspondence – imprimitivity and homogeneous normal

If (S, U, ρ) is an imprimitivity for some compact set S, then the d-tuple $(\rho(z_1), \ldots, \rho(z_d))$ of commuting normal operators is homogeneous by definition with $\sigma(\rho(z_1), \ldots, \rho(z_d)) = S$. The other way round, the theorem below shows that if $\boldsymbol{N}$ is a d- tuple of homogeneous normal operators with associated representation U, then $(\sigma(\boldsymbol{N}), U, \rho_{\boldsymbol{N}})$ is an imprimitivity.

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Theorem

Let $\boldsymbol{N} := (N_1, \ldots, N_d)$ be a d- tuple of commuting normal operators defined on a complex separable Hilbert space $\mathcal{H}$. Assume that $\boldsymbol{N}$ is homogeneous under the action of a group G with associated representation U. Then $(\sigma(\boldsymbol{N}), U, \rho_{\boldsymbol{N}})$ is an imprimitivity.

locally compact and transitive

Theorem

Suppose that $\mathbb{S}$ is a locally compact transitive G- space and the action of G extends to $\bar{\mathbb{S}}$, the closure of $\mathbb{S}$ with $g \cdot \partial\mathbb{S} \subseteq \partial\mathbb{S}$.

1. *If $(\mathbb{S}, U, P)$ is an imprimitivity, then there exists a unique spectral measure $\hat{P}$ defined on the Borel σ- algebra $\mathcal{B}$ of $\bar{\mathbb{S}}$ satisfying the imprimitivity condition with $\hat{P}(E) = P(E)$ for every Borel subset E of $\mathbb{S}$. Moreover, $\text{supp}(P) = \bar{\mathbb{S}}$.*
2. *If $(\mathbb{S}, U, P)$ is an imprimitivity, then it defines uniquely a homogeneous commuting tuple of normal operators $\mathbf{N}$ such that $\sigma(\mathbf{N}) = \text{supp}(\hat{P}) = \bar{\mathbb{S}}$, where $\hat{P}$ is the spectral measure of $\mathbf{N}$.*

main theorem on decomposition of imprimitivity

Theorem

Let $\mathbf{N}$ be a homogeneous d- tuple of commuting normal operators acting on some Hilbert space $\mathcal{H}$ and let $S := \sigma(\mathbf{N})$ be the spectrum of $\mathbf{N}$. Assume that S is a G- space and that $S = \cup_{j=0}^{r} S_j$, where each S_j is a G- orbit, therefore pairwise disjoint. Then the imprimitivity $(S, U, \rho_{\mathbf{N}})$ induced by $\mathbf{N}$ is equivalent to the imprimitivity $(S, \pi_\mu, \hat{U})$, i.e., there is a unitary

$$\Gamma : \mathcal{H} \to \oplus L^2(E_n, \mu : \mathcal{H}_n)$$

such that $\Gamma \rho(f) \Gamma^* = \pi_\mu(f), f \in C(S)$ and $\Gamma U \Gamma^* = \hat{U}$ is a multiplier representation.

Corollary

Let $\boldsymbol{N}$ be a homogeneous d- tuple of commuting normal operators acting on some Hilbert space $\mathcal{H}$ and let $S := \sigma(\boldsymbol{N})$ be the spectrum of $\boldsymbol{N}$. Assume that $S = \cup_{j=0}^{r} S_j$, where each S_j is a G- orbit and is not necessarily compact. Then there exist quasi-invariant measures μ_j living on S_j such that $\boldsymbol{N}$ is unitarily equivalent to the direct sum of $\boldsymbol{M}^{(j)}$ of the multiplication by the coordinate functions acting on the Hilbert space $L^2(S_j, \mu_j, \mathcal{H}_{n_j}), \dim(\mathcal{H}_{n_j}) = n_j, 0 \leqslant j \leqslant r$.

We point out that $\mathcal{H}_{n_j}$ may be isomorphic to $\mathcal{H}_{n_k}$ even if $j \neq k$.

more examples, the product domains

- Suppose that $\sigma(N) = \bar{\mathbb{D}} \times \bar{\mathbb{D}}$. The subset $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ of $\mathbb{C}^2$ is a G-space, where G consists of pairs $\phi := (\phi_1, \phi_2)$, where ϕ_1, ϕ_2 are Möbius maps of the unit disc. The automorphism ϕ extends to an automorphism of $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ with $\phi(\partial\mathbb{D}^2) \subseteq \partial\mathbb{D}^2$.

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- To identify homogeneous (under the G- action) pairs of commuting normal operators, we first note that the spectrum of such a pair must be a G- invariant compact subset of $\mathbb{C}^2$. To find these, note that the orbit through a point $(z_1, z_2) \in \mathbb{T} \times \mathbb{D}$ is $\mathbb{T} \times \mathbb{D}$, similarly, $\mathbb{D} \times \mathbb{T}$ is also a G- orbit. If $(z_1, z_2) \in \mathbb{T} \times \mathbb{T}$, the G- orbit is $\mathbb{T} \times \mathbb{T}$. These are all the G- orbits in the boundary of $\mathbb{D} \times \mathbb{D}$.

more examples, the product domains

- Suppose that $\sigma(N) = \bar{\mathbb{D}} \times \bar{\mathbb{D}}$. The subset $\mathbb{D}^2 := \mathbb{D} \times \mathbb{D}$ of $\mathbb{C}^2$ is a G-space, where G consists of pairs $\phi := (\phi_1, \phi_2)$, where ϕ_1, ϕ_2 are Möbius maps of the unit disc. The automorphism ϕ extends to an automorphism of $\bar{\mathbb{D}} \times \bar{\mathbb{D}}$ with $\phi(\partial\mathbb{D}^2) \subseteq \partial\mathbb{D}^2$.
- To identify homogeneous (under the G- action) pairs of commuting normal operators, we first note that the spectrum of such a pair must be a G- invariant compact subset of $\mathbb{C}^2$. To find these, note that the orbit through a point $(z_1, z_2) \in \mathbb{T} \times \mathbb{D}$ is $\mathbb{T} \times \mathbb{D}$, similarly, $\mathbb{D} \times \mathbb{T}$ is also a G- orbit. If $(z_1, z_2) \in \mathbb{T} \times \mathbb{T}$, the G- orbit is $\mathbb{T} \times \mathbb{T}$. These are all the G- orbits in the boundary of $\mathbb{D} \times \mathbb{D}$.
- Closure of these obits gives us compact sets that are G-invariant. Moreover, if (z_1, z_2) is in $\mathbb{D}^2$, then the G- orbit through this point is $\mathbb{D}^2$. Thus, all the compact G- invariant subset of $\mathbb{C}^2$ are

$$\bar{\mathbb{D}} \times \bar{\mathbb{D}},\ \mathbb{T} \times \bar{\mathbb{D}},\ \bar{\mathbb{D}} \times \mathbb{T},\ \mathbb{T} \times \mathbb{T}.$$

explicit description

- Among these, the group G acts transitively only on $\mathbb{T} \times \mathbb{T}$. Consequently, pairs $\boldsymbol{N}$ of homogeneous normal operators with $\sigma(\boldsymbol{N})$ are described by Mackey's theorem. We now explicitly describe the remaining three cases.

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- If we consider a commuting pair of homogeneous normal operators $\boldsymbol{N}$ with $\sigma_{\boldsymbol{N}} = \bar{\mathbb{D}} \times \bar{\mathbb{D}}$, then it must be unitarily equivalent to the pair of multiplication operators $\boldsymbol{M} = (M_1, M_2)$ acting on $L^2(\bar{\mathbb{D}} \times \bar{\mathbb{D}}, \mu, \mathcal{H}_n)$, where μ is quasi-invariant with respect to the group G and $\dim \mathcal{H} = n$. The restriction of the measure μ to the transitive G- space $\mathbb{D} \times \mathbb{D}, \mathbb{D} \times \mathbb{T}, \mathbb{T} \times \mathbb{D}$ and $\mathbb{T} \times \mathbb{T}$ is uniquely determined since the group acts on these transitively.

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- These are the measures: $\mu_1 := dA \times dA$, $\mu_2 := dA \times d\theta$, $\mu_3 = d\theta \times dA$ and $\mu_4 := d\theta \times d\theta$, respectively. (Here, dA and $d\theta$ denote the area and the arc length measure, respectively.)

putting them all together

Evidently, $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$. Moreover, μ_i, $1 \leqslant i \leqslant 4$, are mutually singular. Consequently, $L^2(\bar{\mathbb{D}} \times \bar{\mathbb{D}}, \mu, \mathcal{H}_n)$ must be a direct sum of the form

$$L^2(\mathbb{D} \times \mathbb{D}, \mu_1, \mathcal{H}_{n_1}) \oplus L^2(\mathbb{D} \times \mathbb{T}, \mu_2, \mathcal{H}_{n_2}) \\ \oplus L^2(\mathbb{T} \times \mathbb{D}, \mu_3, \mathcal{H}_{n_3}) \oplus L^2(\mathbb{T} \times \mathbb{T}, \mu_4, \mathcal{H}_{n_4}),$$

where $n = n_1 + n_2 + n_3 + n_4$.

Thank You!