Spherical operators

School of Mathematics
120th anniversary of the Hebai normal university

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joint with
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Indian Statistical Institute
Bangalore
And
Indian Institute of Technology
Gandhinagar

what we do

$\mathscr{U}(d)$-homogeneous

Let $\mathbb{B}_d$ be the open Euclidean ball in $\mathbb{C}^d$ and $\boldsymbol{T} := (T_1, \ldots, T_d)$ be a commuting tuple of bounded linear operators on a complex separable Hilbert space $\mathscr{H}$.

Let $\mathscr{U}(d)$ be the linear group of unitary transformations acting on $\mathbb{C}^d$ by the rule: $\boldsymbol{z} \mapsto u \cdot \boldsymbol{z}$, $\boldsymbol{z} \in \mathbb{C}^d$, where $u \cdot \boldsymbol{z}$ is the usual matrix product.

Let $u_1(\boldsymbol{z}), \ldots, u_d(\boldsymbol{z})$ be the coordinate functions of $u \cdot \boldsymbol{z}$. We define $u \cdot \boldsymbol{T}$ to be the operator $(u_1(\boldsymbol{T}), \ldots, u_d(\boldsymbol{T}))$ and say that $\boldsymbol{T}$ is $\mathscr{U}(d)$-homogeneous if $u \cdot \boldsymbol{T}$ is unitarily equivalent to $\boldsymbol{T}$ for all $u \in \mathscr{U}(d)$.

What are all the $\mathscr{U}(d)$-homogeneous tuples $\boldsymbol{M}$ of multiplication by coordinate functions on a Reproducing Kernel Hilbert space $\mathscr{H}_K(\mathbb{B}_d, \mathbb{C}^n) \subseteq \mathrm{Hol}(\mathbb{B}_d, \mathbb{C}^n)$, where n is the dimension of the joint kernel of the d-tuple $\boldsymbol{M}^*$.

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The case of $n=1$ is well understood. The kernel function K in this case is of the form:

$$K(\boldsymbol{z},\boldsymbol{w}) = \sum_{n\geq 0} a_k \langle \boldsymbol{z},\boldsymbol{w}\rangle^k,\ \boldsymbol{z},\boldsymbol{w}\in\mathbb{B}_d,\ a_k>0.$$

In this talk, we focus on the case of $n=d$.

We describe a large class of $\mathscr{U}(d)$-homogeneous operators for $n=d$ and obtain explicit criterion for (i) boundedness, (ii) reducibility and (iii) mutual unitary equivalence of these operators.

We classify the kernels K taking values in $\mathscr{M}_\ell(\mathbb{C})$, $1\leq \ell\leq d$, quasi-invariant under an irreducible unitary representation c of the group $\mathscr{U}(d)$. A crucial ingredient of this proof is that the group $SU(d)$ has exactly two irreducible unitary representations (irrurep) of dimension d that are inequivalent and none in dimensions $2,\ldots,d-1$, $d\geq 3$.

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some generalities

Let Ω be a bounded domain in $\mathbb{C}^d$ and G be a topological group (continuous) acting on Ω. This means that there is a map $\gamma : G \times \Omega \to \Omega$ such that $\gamma_g : w \to g \cdot w$ is continuous and $g \to \gamma_g$ is a homomorphism.

A commuting d-tuple M of multiplication by the coordinate functions on a reproducing kernel Hilbert space $(\mathscr{H}, K)$ is said to be G-homogeneous if $g \cdot M$ is unitarily equivalent to M by a fixed unitary U depending on g.

We restrict to the situation where γ_g defines a holomorhic function on the closure $\overline{\Omega}$. In this case, $\gamma_g := (\gamma_g^1, \ldots, \gamma_g^d)$, where each γ_g^i is a holomorphic function on Ω. We set $\gamma_g \cdot M$ to be the operator $(\gamma_g^1(M), \ldots, \gamma_g^d(M))$, where $\gamma_g^i(M)$ is defined by the usual holomorphic functional calculus.

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Thus homogeneity of M amounts to asking for the existence of unitary operators U_g such that

$$(U_g^* M_1 U_g, \ldots, U_g^* M_d U_g) = (\gamma_g^1(M), \ldots, \gamma_g^d(M)),\ g \in G.$$

Example: The simplest example occurs by setting $\Omega = \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, G to be the group hl$\mathrm{Möb}$ of Möbius transformations: $g_{\theta,a}(z) = e^{\imath\theta}\frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$ and finally $M = M_z$ on the reproducing kernel Hilbert spaces $\mathbb{A}^{(\lambda)}$ determined by the reproducing kernel $K^{(\lambda)}(z,w) = \frac{1}{(1-\bar{w}z)^\lambda}$.

In this example, the intertwining unitary U_g, g in $\mathrm{Möb}$, is the operator

$$(U_{g^{-1}} f)(z) = {g'}^{\lambda/2}(z)(f \circ g)(z),\ f \in \mathbb{A}^{(\lambda)},$$

moreover, it is easily verified that $U_g^* M U_g = g \cdot M$, where M is the multiplication by z on the Hilbert space $\mathbb{A}^{(\lambda)}$.

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more examples

In the previous example, the action of the group Möb on the disc $\mathbb{D}$ is transitive and we raised the question of examining homogeneity under the much smaller subgroup of rotations. Let us discuss this in the slightly more general situation of the Euclidean ball.

So, we have the following data: $\Omega = \mathbb{B}_d$, the bi-holomorphic automorphism group G of the ball $\mathbb{B}_d$ (believe it or not, it is really like the group Möb!), the reproducing kernel Hilbert space $\mathbb{A}^{(\lambda)}$ determined by the kernel function $K^{(\lambda)}(z,w) = \frac{1}{(1-\langle z,w\rangle)^\lambda}$, $z,w \in \mathbb{B}_d$.

Now the commuting tuple of operators $(M_1,\ldots,M_d)$ is homogeneous under the automorphism group G of $\mathbb{B}_d$ and the intertwining unitary operator is

$$(U_{g^{-1}}f)(z) = \det\big(Dg\big)(z)^{\lambda/2}(f\circ g)(z),\; f \in \mathbb{A}^{(\lambda)}.$$

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question

The group G acts transitively on the ball $\mathbb{B}_d$, what about finding all d-tuples M acting on a reproducing kernel Hilbert space $(\mathscr{H}, K)$ that are homogeneous under the smaller group $\mathscr{U}(d) \subset G$?

The answer: If we are looking at reproducing kernel Hilbert spaces consisting of scalar valued holomorphic functions, that is, $\mathscr{H} \subset \mathrm{Hol}(\mathbb{B}_d, \mathbb{C})$, then the answer is that K must be of the form

$$K(z,w) = \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n,\ z, w \in \mathbb{B}_d,$$

$a_0, a_1, a_2, \ldots > 0.$

This follows easily from the Peter-Weyl theorem but breaks down, or perhaps much more complicated when the Hilbert space consists of functions taking values in $\mathbb{C}^n$, $n > 1$.

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handy tools

To determine if an operator, or for that matter, a d-tuple of operators is homogeneous, we need to have tools to determine when M and $g \cdot M$ are unitarily equivalent.

This question is not easy to answer in general unless the operator M is normal, or very close to a normal operator.

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First, an operator $T : \mathbb{H} \to \mathbb{H}$ in the Cowen-Douglas class can be realized as a multiplication operator on a reproducing kernel Hilbert space $\mathscr{H} \subset \mathrm{Hol}(\Omega^*)$ as follows.

The map $E : \mathbb{H} \to \mathrm{Hol}(\Omega^*, \mathbb{C}^n)$

$$E(x)(w) = (\langle x, e_1(\bar{w})\rangle, \ldots, \langle x, e_1(\bar{w})\rangle),\ x \in \mathbb{H},$$

is clearly a $\mathbb{C}^n$- valued holomorphic function defined on $\Omega^* := \{w : \bar{w} \in \Omega\}$.

Transplanting the inner product from $\mathbb{H}$ on the image under E in $\mathrm{Hol}(\Omega^*)$, we make E a unitary. Moreover, E intertwines the operators T and M. The image of E is a reproducing kernel Hilbert space and the kernel is given by the formula

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a model theorem for Cowen-Douglas operators

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the Cowen-Douglas theorem and the Curto-Salinas variation

For simplicity, first assume that the dimension of the eigenspace is 1. Set

$$\mathscr{K}(w) = \Big(\Big(-\frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K(w,w)\Big)\Big)_{i,j=1}^m.$$

In this case, the Cowen and Douglas theorem says that two operators T and $\tilde{T}$ are unitarily equivalent if and only if their curvatures, namely, $\mathscr{K}$ and $\widetilde{\mathscr{K}}$ are equal.

The Cowen-Douglas theorem for $n > 1$ is more complicated and involves the covariant derivatives of the curvature.

On the other hand a variation due to Curto and Salinas says that the two operators are unitarily equivalent if and only if there exists a non-vanishing holomorphic function ϕ defined on Ω with the property:

$$K(z,w) = \phi(z)\widetilde{K}(z,w)\overline{\phi(w)}^{\dagger}.$$

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Here the dimension of the eigenspace need not be assumed to be 1.

We can determine when T is unitarily equivalent to $g\cdot T$ in two steps using either the curvature invariant (assuming $n=1$) or the kernel function.

First, using the curvature and setting

$$\mathscr{K}_T(w) = \left(\left(-\frac{\partial^2}{\partial w_i \partial \bar{w}_j}\log K(w,w)\right)\right)_{i,j=1}^{m},$$

the change of variable formula gives

$$\mathscr{K}_{g\cdot T}(w) = (D(g^{-1})(w))^{\dagger}\mathscr{K}(g^{-1}(w))\overline{(D(g^{-1})(w))}.$$

Now, if $g\cdot T$ is unitarily equivalent to T, by the Cowen-Douglas theorem, we have

$$\mathscr{K}_T(w) = (D(g^{-1})(w))^{\dagger}\mathscr{K}_T(g^{-1}(w))\overline{(D(g^{-1})(w))}.$$

Finally, if the group acts transitively, then picking $g_w : g_w(0) = w$, we have

$$\mathscr{K}_T(w) = (D(g_w^{-1})(w))^{\dagger}\mathscr{K}_T(0)\overline{(D(g_w^{-1})(w))}.$$

Thus the curvature of a homogeneous operator is completely determined form its value at 0 if the action of the group is transitive.

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quasi-invariance of the kernel

Now, to characterize homogeneity using the kernel function, we first note that if the operator T is in the Cowen-Douglas class and is realized as the d- tuple M on the reproducing kernel Hilbert space $(\mathscr{H}, K)$, then $g \cdot T$ is unitarily equivalent to the d-tuple M, but this time on the Hilbert space $(\mathscr{H}, K_g)$, where $K_g(z,w) := K(g^{-1}(z), g^{-1}(w))$. Now, we can apply the Curto-Salinas theorem to show that M is unitarily equivalent to $g \cdot M$ if and only if

$$K(z,w) = c(g,z)K(g^{-1}(z), g^{-1}(w))\overline{c(g,w)}^{\dagger},$$

where $c : G \times \Omega \to \mathrm{GL}_n$ is holomorphic for each fixed $g \in G$.

As before, if the group acts transitively, picking $g_w : g_w(0) = w$, we have the quasi-invariance of K:

$$K(w,w) = c(g_w, w)K(0,0)\overline{c(g_w,w)}^{\dagger},$$

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what else do we get from quasi-invariance

The quasi-invariance of the kernel K says that the linear map U_g, $g\in G$, defined by the rule $U_g f(z) = c(g,z) f\circ g$ is a unitary map that intertwines M and $g\cdot M$. Moreover, $g\mapsto U_g$ is a homomorphism if and only if the function $c: G\times\Omega$ is a cocyle:

$$c(g_1g_2, z) = c(g_2,z)c(g_1, g_2\cdot z),\ g_1, g_2\in G,\ z\in\Omega.$$

Clearly, the cocycle is condition is like the chain rule applied to the composition $g_1\circ g_2$ of two maps g_1, g_2.

From now, we assume all our homogeneous operator are such that the intertwining unitary U_g is a representation of the group, or equivalently, K is quasi-invariant and the c is a cocycle.

Finally, in the example of the ball $\mathbb{B}_d$, we claimed that the multiplication operators on $(\mathscr{H}, K)$ are homogeneous under the unitary group $\mathscr{U}(d)$ if K is of the form $\sum_{k\geq 0} a_k\langle z, w\rangle^k$ and therefore these are invariant kernels.

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not transitive

when the action is not transitive

Let $G \cdot w := \{g \cdot w : g \in G\}$ be the orbit of $w \in \Omega$ under the group G. The group action partitions the set Ω into orbits, so that each element of Ω determines an orbit which is the unique orbit for the action of G on Ω to which it belongs. We denote by Ω/G the set of orbits for the action of G on Ω.

Example: For the action of the unitary group $\mathscr{U}(d)$ on the Euclidean ball $\mathbb{B}_d$, the orbit space is $[0,1)$.

In this case, if the values of the invariant kernel $K(z,w) := \sum_{k\geq 0} a_k \langle z, w\rangle^k$ is known only on the set $[0,1)\times[0,1)$, then it is uniquely determined on $\mathbb{B}_d \times \mathbb{B}_d$ since the function K is real analytic.

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when the action is not transitive, contd.

Let K be a quasi-invariant kernel on Ω and suppose that a group G acts on Ω but not transitively, i.e., X/G is not a singleton. The quasi-invariance of K under G now takes the form

$$K(g_w \cdot w_0, g_w \cdot w_0) = c(g_w, w) K([w],[w]) \overline{c(g_w, w)}^{\dagger},$$

where $w \in [w]$ and $g_w \in G$ is picked such that $g_w(w_0) = w$ for some fixed $w_0 \in [w]$. Here, we have temporarily set $[w] = G \cdot w$.

To find all the operators homogeneous under such a group G, pick an arbitrary non-negative definite real analytic function on $[0,1) \times [0,1)$, find a cocycle c and use the quasi-invariance to obtain K on $\mathbb{B}_d \times \mathbb{B}_d$.

If the group G is compact, then we may assume, without loss of generality, that $c(g_w, w)$ is independent of w. Then the cocycle property shows that $g \to c(g)$ is a unitary representation of the group G.

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simple observations about the transformation rule

Let $\lambda > 0$ be chosen so as to ensure K^λ is non-negative definite. It then follows that $K^{2+\lambda}\mathscr{K}$ is a non-negative kernel taking values in $\mathscr{M}_d$. Set

$$\Gamma(w) := K(\cdot,w) \otimes \bar{\partial} K(\cdot,w) - \bar{\partial} K(\cdot,w) \otimes K(\cdot,w).$$

Note that $\Gamma(w) \in \mathscr{H} \otimes \mathscr{H}$, $w \in \Omega$.

Moreover, a straightforward computation using the reproducing property of K shows that

$$\begin{aligned}\langle \Gamma(w), \Gamma(w)\rangle &= \|\frac{\partial}{\partial w} K(w,w)\|^2 K(w,w) - |\langle \frac{\partial}{\partial w} K(w,w), K(w,w)\rangle|^2 \\ &= K(w,w)^2 \frac{\partial^2}{\partial \bar{w} \partial w} \log K(w,w).\end{aligned}$$

This verifies our claim that $K^{2+\lambda}\mathscr{K}$ is a non-negative definite kernel. We have thus produced (with a similar computation involving d- variables) a non-negative definite kernel taking values in $\mathscr{M}_d$ starting from a scalar valued kernel.

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Example

Suppose that K is a quasi-invariant kernel under a group G, again not necessarily transitive. Now that we know, the curvature $K^{\lambda+2}\mathscr{K}$ is a non-negative definite, it follows from the quasi-invariance of K and the transformation rule for the curvature $\mathscr{K}$ that $K^{\lambda+2}\mathscr{K}$ is a quasi-invariant $d \times d$ matrix valued kernel.

From the transformation rule for $\mathscr{K}$, we infer that the cocycle in this case is $c : u \to u^{\dagger}, u \in \mathscr{U}(d)$.

Now, a direct computation gives the following

$$K^{\lambda+2}(re_1, re_1)\mathscr{K}(re_1, re_1) = \frac{d+1}{(1-r^2)^{t(d+1)+2}} \begin{pmatrix} 1 & 0 \\ 0 & (1-r^2)I_{d-1} \end{pmatrix}, 0 \le r < 1.$$

This function on $[0,1) \times [0,1)$ extends (uniquely) to a non-negative definite quasi-invariant kernel on $\mathbb{B}_d \times \mathbb{B}_d$. These kernels are not invariant but merely quasi-invariant. Are there others?

Answer: There are plenty, we can describe all of them.

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Example

Suppose that K is a quasi-invariant kernel under a group G, again not necessarily transitive. Now that we know, the curvature $K^{\lambda+2}\mathscr{K}$ is a non-negative definite, it follows from the quasi-invariance of K and the transformation rule for the curvature $\mathscr{K}$ that $K^{\lambda+2}\mathscr{K}$ is a quasi-invariant $d\times d$ matrix valued kernel.

From the transformation rule for $\mathscr{K}$, we infer that the cocycle in this case is $c : u \to u^{\dagger}, u \in \mathscr{U}(d)$.

Now, a direct computation gives the following

$$K^{\lambda+2}(re_1,re_1)\mathscr{K}(re_1,re_1) = \frac{d+1}{(1-r^2)^{t(d+1)+2}}\begin{pmatrix} 1 & 0 \\ 0 & (1-r^2)I_{d-1}\end{pmatrix}, 0 \le r < 1.$$

This function on $[0,1)\times[0,1)$ extends (uniquely) to a non-negative definite quasi-invariant kernel on $\mathbb{B}_d \times \mathbb{B}_d$. These kernels are not invariant but merely quasi-invariant. Are there others?

Answer: There are plenty, we can describe all of them.

what we have proved

Let $K:\mathbb{B}_d\times\mathbb{B}_d\to\mathscr{M}_n(\mathbb{C})$ be a non-negative definite kernel quasi-invariant under $\mathscr{U}(d)$ with respect to a multiplier c, where $c:\mathscr{U}(d)\to\mathrm{GL}_n(\mathbb{C})$ is an irreducible unitary representation. (a) If $n=d$, then up to unitary equivalence $K(z,w)$ is either of the form

$$\sum_{\ell=1}^{\infty}\left(a_{\ell,1}-a_{\ell,2}\right)\langle z,w\rangle^{\ell-1}\bar{w}z^{\dagger}+\sum_{\ell=0}^{\infty}a_{\ell,2}\langle z,w\rangle^{\ell}I_d,\quad z,w\in\mathbb{B}_d$$

$$a_{\ell,1}\geq 0 \text{ and } a_{\ell,1}\leq(\ell+1)a_{\ell,2},\quad \ell\in\mathbb{Z}_+$$

or of the form

$$\sum_{\ell=1}^{\infty}\left(\tilde{a}_{\ell,1}-\tilde{a}_{\ell,2}\right)\langle z,w\rangle^{\ell-1}z\bar{w}^{\dagger}+\sum_{\ell=0}^{\infty}\tilde{a}_{\ell,2}\langle z,w\rangle^{\ell}l_d,\quad z,w\in\mathbb{B}_d,$$

$$\tilde{a}_{\ell,2}\geq 0 \text{ and } (d-1)\tilde{a}_{\ell,2}\leq(\ell+d-1)\tilde{a}_{\ell,1},\quad \ell\in\mathbb{Z}_+.$$

If $1<n<d$, then there is no n-dimensional irreducible unitary representation c such that K is quasi-invariant under $\mathscr{U}(d)$ with cocycle c.

Thank You!