Induced representations, Mackey Imprimitivity and Homogeneous operators

Gadadhar Misra

2025 March 07

Gadadhar Misra
Indian Statistical Institute Bangalore and Indian Institute of Technology Gandhinagar

Imprimitivity

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- Let G be a locally compact second countable group and $\mathbb{S}$ be a locally compact topological G- space. Let $C_0(\mathbb{S})$ be the algebra of continuous functions vanishing at ∞ on $\mathbb{S}$ and $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$.

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- Let G be a locally compact second countable group and $\mathbb{S}$ be a locally compact topological G- space. Let $C_0(\mathbb{S})$ be the algebra of continuous functions vanishing at ∞ on $\mathbb{S}$ and $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$.
- Suppose that φ is a $*$- homomorphism from $C_0(\mathbb{S})$ into $\mathcal{L}(\mathcal{H})$ and U is a unitary representation of the group G on the same Hilbert space $\mathcal{H}$. Then the imprimitivity is the relationship

$$U(g)\varphi(f)U(g)^* = \varphi(g \cdot f), \; g \in G, \; f \in C_0(\mathbb{S}),$$

where $g \cdot f$ is the function: $(g \cdot f)(s) = f(g^{-1} \cdot s)$, $s \in \mathbb{S}$.

imprimitivity theorem of Mackey

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  $$\big(U(g)h\big)(s) = c(g, s)(g \cdot h)(s),\ \ h \in L^2(\mathbb{S}, \mu, \mathcal{H}_n),\ \ g \in G,$$

  where $c : G \times \mathbb{S} \to \mathcal{U}(\mathcal{H}_n)$ is a Borel map taking values in the group of unitary operators $\mathcal{U}(\mathcal{H}_n)$ of the Hilbert space $\mathcal{H}_n$ of dimension n.

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- For U to be a homomorphism, the function c must be a cocycle.
- The second part of the imprimitivity theorem asserts that such a multiplier representation is induced from a unitary representation of the subgroup H acting on the Hilbert space $\mathcal{H}_n$.

Finding invariant measures

- Let $\mathcal{D}$ be a bounded open connected set in $\mathbb{C}^d$ and dV be the Lebesgue volume measure. Let $L^2_{\mathrm{hol}}(\mathcal{D}, dV)$ be the space of square integrable (wrt dV) holomorphic functions on $\mathcal{D}$. This is a closed subspace of $L^2(\mathcal{D}, dV)$ and therefore a Hilbert space on its own right.

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- The evaluation map $e_z : L^2_{\mathrm{hol}}(\mathcal{D}, dV) \to \mathbb{C}$ is bounded:
  For $z_0 \in \mathcal{D}$, choose a polydisc $P_r(z_0) = \prod_{k=1}^{d} D_r(z_{0,k}) \subset \mathcal{D}$. By the (iterated) mean value property for holomorphic functions:

$$f(z_0) = \frac{1}{(\pi r^2)^d} \int_{P_r(z_0)} f(z)\, dV(z).$$

  Applying the Cauchy-Schwarz inequality, we have

$$|f(z_0)| \le \frac{1}{(\pi r^2)^d} \Big(\int_{P_r(z_0)} |f(z)|^2\, dV(z) \Big)^{1/2} \Big(\int_{P_r(z_0)} 1\, dV(z) \Big)^{1/2}$$

  The second integral equals $(\pi r^2)^d$, so we have

$$|f(z_0)| \le \frac{1}{(\pi r^2)^{d/2}} \|f\|_{L^2(P_r(z_0))} \le \frac{1}{(\pi r^2)^{d/2}} \|f\|_{L^2_{\mathrm{hol}}(\mathcal{D},\, dV)}.$$

The Bergman kernel

- The Bergman kernel $B_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ is defined as $B(z, w) = e_w e_z^*$. It clearly has the reproducing property: $\langle f, B(\cdot, w) \rangle = f(w)$. This follows from the observation that

  $$\langle f, e_z^*(1) \rangle_H = e_z(f) = f(z), \ f \in H.$$

  By the Riesz representation theorem, there exists a unique vector $B_z \in H$ such that: $e_z(f) = \langle f, B_z \rangle_H$ for all $f \in H$. This vector is denoted $B(\cdot, z) \equiv B_z$. Thus,

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- Applying the change of variable formula for integration to the equality below

  $$B(z,w) = \langle B(\cdot, w), B(\cdot, z)\rangle = \int_{\mathcal{D}} B(\cdot, w)\overline{B(\cdot, z)}dV,$$

  we obtain the transformation rule for the Bergman kernel, namely,

  $$B(g\cdot z, g\cdot w) = J_g(z)^{-1}B(z,w)\overline{J_g(w)}^{-1}.$$

Invariant measure

- Set $z = w$: $B(g \cdot z, g \cdot w) = |J_g(z)|^{-2} B(z,z)$. To show that $B(z,z)dV(z)$ is an invariant measure, we must verify

  $$\int_{\mathcal{D}} f(g \cdot z) B(z,z) dV(z) = \int_{\mathcal{D}} f(z) B(z,z) dV(z).$$

  Using change of variable $z = g^{-1} \cdot u$, $dV(z) = |J_{g^{-1}}(u)|^2 dV(u)$.

  $$\int_{\mathcal{D}} f(g \cdot z) B(z,z) dV(z) = \int_{\mathcal{D}} f(u) B(g^{-1} \cdot u, g^{-1} \cdot u) |J_{g^{-1}}(u)|^2 dV(u).$$

  We have $B(g^{-1} \cdot u, g^{-1} \cdot u) = |J_{g^{-1}}(u)|^{-2} B(u,u)$. Therefore,

  $$\begin{aligned} \int_{\mathcal{D}} f(u) B(g^{-1} \cdot u, g^{-1} \cdot u) |J_{g^{-1}}(u)|^2 dV(u) &= \int_{\mathcal{D}} f(u) B(u,u) dV(u) \\ &= \int_{\mathcal{D}} f(z) B(z,z) dV(z). \end{aligned}$$

  Thus, $B(z,z)dV(z)$ is an invariant measure.

Finding quasi-invariant measure

- Let $\mathcal{D}$ be an open bounded connected subset of $\mathbb{C}^d$ and B be the Bergman kernel of $\mathcal{D}$. Write $j(g, z) := J_g(z)^{-1}$ for the inverse complex Jacobian (a cocycle, by the chain rule). Let $\mu = B(z, z)dV(z)$ be the invariant measure, and set

$$\boxed{\mu_t := B(z, z)^{-t} B(z, z) dV(z) = B(z, z)^{-t} d\mu(z),}$$

Claim: μ_t is quasi-invariant, $t \geqslant t_0$.

$$\begin{aligned} \int f(g \cdot z) d\mu_t(z) &= \int f(g \cdot z) B(z, z)^{-t} d\mu(z) \\ &= \int f(w) B\left(g^{-1} w, g^{-1} \cdot w\right)^{-t} d\mu(w), \ w = g \cdot z. \\ &= \int f(w) j\left(g^{-1}, w\right)^{-t} B(w, w)^{-t} \overline{j(g^{-1}, w)^{-t}} d\mu(w) \\ &= \int f(w) \left| j\left(g^{-1}, w\right)\right|^{-2t} B(w, w)^{-t} d\mu(w) \\ &= \int f(w) |j(g^{-1}, w)|^{-2t} d\mu_t(w). \end{aligned}$$

Unitary representation

- Thus the Radon-Nikodym derivative of μ_t is $|j\left(g^{-1}, w\right)|^{-2t}$ and $U_g : \mathbb{A}^{(t)}(\mathcal{D}, \mu_t) \to \mathbb{A}^{(t)}(\mathcal{D}, \mu_t)$ defined by the formula

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- However, the question that remains is when do we have quasi-invariant measure on G/H and a measurable cross-section from G/H to G. While this is relatively speaking not hard to answer if G is a Lie group and $H \subset G$ is a closed subgroup, the general case, of a locally compact second countable group is much more complicated. However, even in that case, both questions have affirmative answers thus providing a satisfactory theory of unitarily induced representations in this case.

Example

- We let $G^{(0)}$ denote the Möbius group of biholomorphic automorphisms of the complex open unit disc $\mathbb{D}$, and let $G^{(1)}$ be the group of 2×2 matrices of the form

  $$g^{(1)} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \ |a|^2 - |b|^2 = 1.$$

  As is well known, $G^{(1)}$ acts on $\mathbb{D}$ as

  $$g^{(1)}(z) = g^{(1)} \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}, \ z \in \mathbb{D}.$$

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- Each $g_{\theta,\alpha} \in G^{(0)}$ is of the form $k_\theta^{(0)}\, p_\alpha^{(0)}$, where $k_\theta^{(0)} z = e^{i\theta} z$, $\theta \in \mathbb{R} \mod 2\pi$, and

  $$p_\alpha^{(0)}(z) := \frac{\alpha - z}{1 - \bar{\alpha} z}, \ \alpha \in \mathbb{D},$$

  is an involution mapping 0 to α.

The covering groups

- All elements of $G^{(0)}$ arise from two elements of $G^{(1)}$, namely, $g^{(1)}$ and $-g^{(1)}$. This gives a homomorphism $\pi_0^1 : G^{(1)} \to G^{(0)}$. With the natural topology on $G^{(0)}$, this is also a covering; so $G^{(1)}$ is a covering group of $G^{(0)}$.

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- The kernel of π_0^1 is the subgroup $M^{(1)} = \{\pm I_2\}$, where I_2 is the 2×2 identity matrix.
- The unitary elements of $G^{(1)}$ form a subgroup $K^{(1)}$. They are

$$k_\theta^{(1)} = \begin{pmatrix} e^{i(\theta/2)} & 0 \\ 0 & e^{-i(\theta/2)} \end{pmatrix}, \ \theta \in \mathbb{R} \mod 4\pi.$$

The positive definite Hermitian elements are

$$p_\alpha^{(1)} = (1 - |\alpha|^2)^{-1/2} \begin{pmatrix} 1 & \alpha \\ \bar{\alpha} & 1 \end{pmatrix}, \ \alpha \in \mathbb{D}.$$

They form a connected set $P^{(1)}$.

The universal covering group

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- The elements of the universal covering group G can be uniquely written as $k_\theta \, p_\alpha$, where now k_θ is parameterized by $\theta \in \mathbb{R}$. Set $K := \{k_\theta \mid \theta \in \mathbb{R}\}$, $K \cong \mathbb{R}$. The covering homomorphism $\pi_0 : G \to G^{(0)}$ is given by

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- The *action of* G *on* $\overline{\mathbb{D}}$ is defined by $g(z) := (\pi_1 \circ g)(z)$, $z \in \overline{\mathbb{D}}$, where $\pi_1 : G \to G^{(1)}$ is the covering homomorphism; equivalently $g(z) = (\pi_0 \circ g)(z)$, since $\pi_0 = \pi_0^1 \circ \pi_1$. The stabilizer of 0 can be shown to be K.

Discrete series representations

- To find the representation induced by the character $\rho_n : K^{(1)} \to \mathbb{T}$ with $\rho_n(k_\theta^{(1)}) = \left(e^{i\theta/2}\right)^{-2n} = e^{-in\theta}$, $n \in \mathbb{N}$, we have to compute

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- Let us find $k_\theta^{(1)}$ such that

$$g\, p_z = p_{g\cdot z}\, k_\theta^{(1)}, \;\; g \in \mathrm{SU}(1,1).$$

- Choosing $g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$, we have that

$$g\, p_z = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z \\ \bar z & 1 \end{pmatrix} = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} \alpha + \beta\bar z & \alpha z + \beta \\ \bar\beta + \bar\alpha\bar z & \bar\beta z + \bar\alpha \end{pmatrix}.$$

  A straightforward computation gives

$$1 - |g\cdot z|^2 = \frac{1-|z|^2}{|\bar\beta z + \bar\alpha|^2} = (1-|z|^2)|g'(z)|.$$

Determining the cocycle

- Hence,

$$p_{g\cdot z} = \frac{1}{\sqrt{1-|g\cdot z|^2}} \begin{pmatrix} 1 & \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}} \\ \frac{\overline{\alpha}\bar{z}+\overline{\beta}}{\beta\bar{z}+\alpha} & 1 \end{pmatrix} = \frac{|\overline{\beta} z+\overline{\alpha}|}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}} \\ \frac{\overline{\alpha}\bar{z}+\overline{\beta}}{\beta\bar{z}+\alpha} & 1 \end{pmatrix}$$

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- Taking

$$k_\theta^{(1)} = \begin{pmatrix} \frac{\alpha+\beta\bar z}{|\bar\beta z+\bar\alpha|} & 0 \\ 0 & \frac{\bar\alpha+\bar\beta z}{|\bar\beta z+\bar\alpha|} \end{pmatrix},$$

  we have $g\, p_z = p_{g\cdot z}\, k_\theta^{(1)}$. Therefore,

$$(p_{g\cdot z})^{-1}\, g\, p_z = k_\theta^{(1)},$$

  and it follows that

$$\rho_n\big((p_{g\cdot z})^{-1}\, g\, p_z\big) = \left(\frac{\bar\alpha+\bar\beta z}{|\bar\beta z+\bar\alpha|}\right)^{2n} = \left(\frac{g'(z)}{|g'(z)|}\right)^{-n}.$$

The explicit determination of the representation

- We can directly verify that $m_{\rho_n} : \mathrm{SU}(1,1) \times \mathbb{D} \to \mathbb{C} \setminus \{0\}$, given by the formula

  $$m_{\rho_n}(g,z) := \rho_n\big((p_{g\cdot z})^{-1}\, g\, p_z\big)$$

  is a multiplier for each $n \in \mathbb{N}$. We can therefore define a unitary representation $U : \mathrm{SU}(1,1) \to \mathcal{U}(L^2(\mathbb{D}, d\mu))$, where μ is the invariant measure $(1-|z|^2)^{-2} dA$, by setting

  $$\begin{aligned} U_g f(z) &= m_{\rho_n}(g, g^{-1} \cdot z)(f \circ g^{-1})(z) \\ &= \tfrac{|g'(g^{-1}\cdot z)|^n}{g'(g^{-1}\cdot z)^n}(f \circ g^{-1})(z), \quad g \in \mathrm{SU}(1,1). \end{aligned}$$

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- We can write the representation U in an equivalent form acting on $L^2(\mathbb{D}, (1 - |z|^2)^{2n} d\mu)$ by conjugating with the unitary operator

  $$\Gamma : L^2(\mathbb{D}, d\mu) \to L^2(\mathbb{D}, (1 - |z|^2)^{2n} d\mu)$$

  defined by setting $(\Gamma f)(z) = (1 - |z|^2)^{-n} f(z)$.

The representations in the standard form

- Let us compute $\Gamma U_g \Gamma^*$:

$$\begin{aligned}(\Gamma U_g \Gamma^* f)(z) &= (1-|z|^2)^{-n}(U_g \Gamma^* f)(z)\\ &= (1-|z|^2)^{-n}\left(\tfrac{g'(g^{-1}\cdot z)}{|g'(g^{-1}\cdot z)|}\right)^{-n}(\Gamma^* f)(g^{-1}\cdot z)\\ &= (1-|z|^2)^{-n}\left(\tfrac{g'(g^{-1}\cdot z)}{|g'(g^{-1}\cdot z)|}\right)^{-n}(1-|g^{-1}\cdot z|^2)^{n}(f\circ g^{-1})(z).\end{aligned}$$

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$$\boxed{(\Gamma U_g \Gamma^* f)(z) = \big(g'(g^{-1}\cdot z)\big)^{-n}(f\circ g^{-1})(z).}$$

- The representation $\Gamma U_g \Gamma^*$ acting on $L^2(\mathbb{D}, (1-|z|^2)^{2n} d\mu)$ is reducible. Restricting to the reducing subspace of holomorphic functions yields an irreducible unitary representation.
- Since the derivative of g, regarded as a function on $\mathbb{D}$, is the square of a holomorphic function on $\mathbb{D}$, the multiplier $(g' \circ g^{-1})^{-n}$ is single-valued and continuous on $\mathrm{SU}(1,1) = G^{(1)}$ if

Thank You!