Induced representations, Mackey Imprimitivity and Homogeneous operators

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Homogeneous vector bundles

- Let G be a group and $H \subset G$ be a subgroup. Then G acts transitively on $X = G/H$, the left coset space (H can be identified with the stabilizer group G_0 of any $x_0 \in X$). Suppose that $\rho : H \to \mathrm{GL}(V)$ is a representation. Then from this data, we construct a (homogeneous) vector bundle as follows:

$$E^\rho = G \times_H V = (G \times V)/\sim,$$

where $\sim$ is defined by $(g, v) \sim \left(gh^{-1}, \rho(h)v\right)$ for all $h \in H$.

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  where $\sim$ is defined by $(g, v) \sim \left(gh^{-1}, \rho(h)v\right)$ for all $h \in H$.
- The projection map $\pi : E^\rho \to G/H$ is given by $\pi([g, v]) = gH$. For $[g, v_1]$ and $[g, v_2]$ in the same fiber, define $[g, v_1] + [g, v_2] = [g, v_1 + v_2]$ and $c[g, v] = [g, cv]$. This is well-defined:

$$\begin{aligned}[gh^{-1}, \rho(h)v_1] + [gh^{-1}, \rho(h)v_2] &= [gh^{-1}, \rho(h)v_1 + \rho(h)v_2] \\ &= [gh^{-1}, \rho(h)(v_1 + v_2)] = [g, v_1 + v_2],\end{aligned}$$

$$c[gh^{-1}, \rho(h)v] \;=\; [gh^{-1}, c\rho(h)v] \;=\; [gh^{-1}, \rho(h)(cv)] \;=\; [g, cv].$$

The fiber is isomorphic to V

- We verify that each of the fibers over a point $gH \in G/H$ is isomorphic to V.
  Since $gh^{-1} \in gH$, it follows that both gh_1^{-1}, gh_2^{-1} are in the same coset gH.
  Let $\phi : V \to E^{\rho}_{gH}$ be the map $\phi(v) = [g, v]$. Check that ϕ is linear and bijective:

$$\phi(v_1 + v_2) = [g, v_1 + v_2] = [g, v_1] + [g, v_2] = \phi(v_1) + \phi(v_2)$$
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- Moreover, ϕ is independent of the representative: Choosing gh^{-1} instead of g, the map ϕ takes the form $\hat{\phi}(w) = [gh^{-1}, w]$ which is related to ϕ via the linear transformation ρ, namely, $w = \rho(h)v$.

cocycles and homogeneous bundles

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- Let us find the action of G in the local trivialization via a cross-section:
  To trivialize the bundle, choose a cross-section $s : G/H \to G$ as follows. As before, given $x \in X$ we have $g = g_x h$ for some $h \in H$, and set $s(x) = g_x$. This allows us to identify:

$$G \times_H V \simeq G/H \times V \quad \text{via} \quad [s(x), v] \leftrightarrow (x, v).$$

  The group G moves x to $g \cdot x$ and simultaneously maps the fiber over x to the fiber over $g \cdot x$. For this, we need to express $g \cdot s(x) \in G$ in terms of the section $s(g \cdot x)$, as before!

Decomposing $g \cdot s(x)$

- Decompose $g \cdot s(x)$: Using the section s, write:

$$g \cdot s(x) = s(g \cdot x) \cdot h(g, x),$$

where $h(g, x) \in H$. This uniquely defines $h(g, x)$ as:

$$h(g, x) = s(g \cdot x)^{-1} \cdot g \cdot s(x).$$

Indeed, we had verified earlier that $s(g \cdot x)^{-1} \cdot g \cdot s(x)$ is in H. Thus, the cocycle $c(g, x)$ is defined using the representation ρ as follows:

$$c(g, x) = \rho(h(g, x)) \in \mathrm{GL}(V).$$

Now, g action on the equivalence class $[s(x), v]$ is easy to compute:

$$g \cdot [s(x), v] = [g \cdot s(x), v] = [s(g \cdot x) \cdot h(g, x), v].$$

- By the equivalence relation in $G \times_H V$:

$$[s(g \cdot x) \cdot h(g,x), v] \sim [s(g \cdot x), \rho(h(g,x))v].$$

Under the trivialization $[s(g \cdot x), \rho(h(g,x))v] \leftrightarrow (g \cdot x, \rho(h(g,x))v)$, this becomes:

$$g \cdot (x, v) = (g \cdot x, c(g,x)v),$$

where $c(g,x) = \rho(h(g,x))$.
- Cocycle Condition: The definition ensures $c(g_1 g_2, x) = c(g_1, g_2 x) \cdot c(g_2, x)$, as required for consistency under group multiplication:

$$\rho(h(g_1 g_2, x)) = \rho(h(g_1, g_2 x) \cdot h(g_2, x)) = \rho(h(g_1, g_2 x)) \cdot \rho(h(g_2, x)).$$

We let E^ρ denote the bundle obtained from the group G, a subgroup $H \subset G$ and a representation $\rho : H \to \mathrm{GL}(V)$ as described above.

Induced representations and homogeneous vector bundles

- Recall that the induced representation $\mathrm{Ind}_H^G(\rho)$ consists of functions $f : G \to V$ satisfying:

  $$f(gh) = \rho\left(h^{-1}\right) f(g) \text{ for all } g \in G, h \in H.$$

  The group G acts on these functions by left translation: $\left(g' \cdot f\right)(g) = f\left(g'^{-1} g\right)$.

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- A section γ of the bundle E^ρ is a map that has the property $\pi \circ \gamma : G/H \to G/H$ is the identity map. Given a section γ of E^ρ, we can define a function $f_\gamma : G \to V$ by setting

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- The other way round, from functions to sections: Given a function $f \in \mathrm{Ind}_H^G(\rho)$, we can define a section γ_f of E^ρ by setting

$$\gamma_f(gH) = [g, f(g)].$$

  If $gH = g'H$, then $g' = gh$ for some $h \in H$, and the equivariance condition ensures that γ_f is well-defined.

Isomorphism of Representations

- The two maps, $f_\gamma(g) = \mathsf{proj}_V\left(g^{-1} \cdot s(gH)\right)$ and $\gamma_f(gH) = [g, f(g)]$ are inverses of each other.

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- Let us define the map $\tau : G \to \mathsf{InvLin}(\Gamma\left(E^\rho\right))$ by setting

$$\tau_g(\gamma)(x) = g \cdot \gamma\left(g^{-1} \cdot x\right),\ \gamma \in \Gamma\left(E^\rho\right).$$

  Under the correspondence described between sections of the vector bundle E^ρ and functions on G, the representation τ is isomorphic to $\mathsf{Ind}_H^G(\rho)$.
  This correspondence respects the group action: For $g' \in G$,

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Unitary representations

- We have seen that $\mathrm{Ind}_{G_0}^{G}(\rho)$ is a homomorphism from G into $\mathrm{InvLin}(\mathcal{F}(X, V))$. We now ask when can we make this induced representation unitary on some subspace of $\mathcal{F}(X, V)$ equipped with the structure of a Hilbert space? An obvious choice of a Hilbert subspace of $\mathcal{F}(X, V)$ is the space $L^2_V(X, \mu)$.

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- In general, if $\phi : X \to Y$ and X is a measure space, then the push-forward of $\phi_*\mu$ is defined on Y by setting

$$\phi_*\mu(A) = \mu(\phi^{-1}(A)),\ A \subseteq Y.$$

Equivalently,

$$\int f \circ \phi \, d\mu = \int f d(\phi_*\mu).$$

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- Now, if μ and $\phi_*\mu$ are mutually absolutely continuous, then

$$d(\phi_*\mu) = \frac{d(\phi_*\mu)}{d\mu} \, d\mu,$$

  where $\frac{d(\phi_*\mu)}{d\mu}$ is the Radon-Nikodym derivative of $\phi_*\mu$ wrt μ.

quasi-invariance gives multipliers

- The quasi-invariance of the measure μ is equivalent to the condition:
$$d(g_*\mu) = \frac{d(g_*\mu)}{d\mu}\, d\mu$$
for all $g \in G$. Now, for $X = Y$ and $g \in G$, we have
$$\int f\, d\mu = \int f \circ g^{-1}\, d(g_*\mu) = \int f \circ g^{-1} \frac{d(g_*\mu)}{d\mu}\, d\mu.$$

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- Define the operator
$$(T_g f)(x) = \left(\frac{d(g_*\mu)}{d\mu}(x)\right)^{1/2} f(g^{-1} \cdot x).$$
Then
$$\|T_g f\|^2 = \int \left|f(g^{-1}\cdot x)\right|^2 \frac{d(g_*\mu)}{d\mu}(x)\, d\mu(x) = \int \left|f(g^{-1}\cdot x)\right|^2 d(g_*\mu)(x) = \int |f|^2$$
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- Therefore, a multiplier representation that is unitary can be made up as soon as we have a quasi-invariant measure.

Thank You!