# Induced representations, Mackey Imprimitivity and Homogeneous operators

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## Group action, Multiplier representation

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  We will generally, write $o$ instead of $x_0$. The stabilizer group $G_0$ of $o$ is the subgroup $\{g \in G \mid g \cdot o = o\}$ of $G$.
- Let $X$ be a set and $\mathcal{F} = \mathrm{Func}(X, \mathbb{C})$ be the set of all functions from $X$ to $\mathbb{C}$, and $\mathbf{Lin}(\mathcal{F})$, be the space of linear maps on $\mathcal{F}$. Let us define a map $T : G \to \mathbf{Lin}(\mathcal{F})$, called multiplier representation:

$$(T_{g^{-1}}f)(x) = c(g,x)^{-1}f(g \cdot x), \;\; g \in G, \;\; x \in X,$$

where $c : G \times X \to \mathbb{C} \setminus \{0\}$ is some function.

# Multipliers

- Suppose that $T$ is a homomorphism. Then we must have $T_{g_1^{-1}} T_{g_2^{-1}} = T_{g_1^{-1} g_2^{-1}} = T_{(g_2 g_1)^{-1}}$. Consequently, we have that

$$
\begin{aligned}
c(g_2 g_1, x)^{-1} f((g_2 g_1) \cdot x) &= \big(T_{(g_2 g_1)^{-1}} f\big)(x) \\
&= \big(T_{g_1^{-1}} T_{g_2^{-1}} f\big)(x) \\
&= T_{g_1^{-1}} \big(T_{g_2^{-1}} f\big)(x) \\
&= c(g_1, x)^{-1} \big(T_{g_2^{-1}} f\big)(g_1 \cdot x) \\
&= \big(c(g_1, x)^{-1} c(g_2, g_1 \cdot x)^{-1} f\big)(g_2 \cdot (g_1 \cdot x)).
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- Thus, we conclude that

$$c(g_2 g_1, x) = c(g_2, g_1 \cdot x)\, c(g_1, x), \;\; g_1, g_2 \in G, \;\; x \in X.$$

Evidently, $c(e, x) = 1$. Such a map $c : G \times X \to \mathbb{C} \setminus \{0\}$ is called a cocycle.
Conversely, if $c$ is a cocycle, then the map $g \mapsto T_g$ is a homomorphism.

# The Stabilizer subgroup

- Fix $o(= x_0) \in X$ and set $c(g) := c(g, o), g \in G$. By transitivity of the group action, given any $x \in X$, there is a $g_x$ such that $g_x \cdot o = x$. Pick any $g \in G$ such that $g \cdot o = x$. Then

$$g_x^{-1}(g \cdot o) = g_x^{-1} \cdot x = o.$$

  In other words, $g_x^{-1}g$ is in the stabilizer subgroup $G_0$, or equivalently, $g = g_x g_0$ for some $g_0 \in G_0$.

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- Therefore, it follows that

$$\begin{aligned} c(gg_x) &= c(gg_x, o) \\ &= c(g, g_x \cdot o)c(g_x, o) \\ &= c(g, x)c(g_x, o) \\ &= c(g, x)c(g_x). \end{aligned}$$

Thus, we have $c(g, x) = c(gg_x)c(g_x)^{-1}$

# Determining the multiplier

- The transitivity of the $G$ action ensures that every $g$ in $G$ is of the form $g_y g_0$ for some $y \in X$ and $g_0 \in G_0$. Since $gg_x$ is in $G$, we can find a $y \in X$ such that $gg_x = g_y g_0$. Note that

$$(gg_x) \cdot o = g \cdot (g_x \cdot o) = g \cdot x.$$

Therefore, it follows that $y = g \cdot x$ and we have

$$\begin{aligned} c(g, x) &= c(gg_x)c(g_x)^{-1} \\ &= c(g_y g_0)c(g_x)^{-1} \\ &= c(g_y g_0, o)c(g_x)^{-1} \\ &= c(g_y, g_0 \cdot o)c(g_0, o)c(g_x)^{-1} \\ &= c(g_y)c(g_0)c(g_x)^{-1} \end{aligned}$$

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- Let $A : \mathcal{F} \to \mathcal{F}$ be the linear map

$$(Af)(x) \;=\; c(g_x)f(x),\;\; x \in X,\;\; g_x \in G; g_x \cdot o = x.$$

# Multiplier representation, again!

- We compute the conjugate of $T_g$ under $A$:

$$\begin{aligned}\big(A^{-1}T_gAf\big)(x) &= c(g_x)^{-1}\big(T_gAf\big)(x)\\ &= c(g_x)^{-1}c(g, g^{-1}\cdot x)(Af)(g^{-1}\cdot x)\\ &= c(g_x)^{-1}c(g, g^{-1}\cdot x)c(g_{g^{-1}\cdot x})f(g^{-1}\cdot x)\\ &= c(g_x)^{-1}c(g_x)c(g_0)c(g_{g^{-1}\cdot x})^{-1}c(g_{g^{-1}\cdot x})f(g^{-1}\cdot x)\\ &= c(g_0)f(g^{-1}\cdot x),\end{aligned}$$

where $g_0 \in G_0$ is determined from the equation $g\, g_{g^{-1}\cdot x} = g_x\, g_0$.

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  where $g_0 \in G_0$ is determined from the equation $g\, g_{g^{-1}\cdot x} = g_x\, g_0$.
- We conclude that $\tilde{T}_g := A^{-1}T_gA$ is given by the formula

$$\big(\tilde{T}_g f\big)(x) = \tilde{c}(g,x)f(g^{-1}\cdot x),$$

  where $\tilde{c}(g,x) = c(g_0)$ with $g_0 = g_x^{-1}g\, g_{g^{-1}\cdot x} \in G_0$; thus $\tilde{c}$ takes its values through the restriction $c|_{G_0}$. Moreover,

$$c(gg_0) = c(gg_0, o) = c(g, g_0\cdot o)c(g_0, o) = c(g)c(g_0).$$

  Thus, $c(g) := c(g,o)$, $g \in G_0$, is a homomorphism of $G_0$.

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# Inducing from a representation

- Suppose that $\rho : G_0 \to \mathrm{GL}(V)$ is a homomorphism. We will define a multiplier representation associated with $\rho$ to be called the representation induced from $\rho$. For this, it is enough to construct a $\mathrm{GL}(V)$- valued cocycle starting from $\rho$.

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- The representation $\mathrm{Ind}_{G_0}^{G}(\rho)$ of the group $G$ induced from the representation $\rho : G_0 \to \mathbf{GL}(V)$ of $G_0 \subset G$ is a multiplier representation of the form

$$\big(\mathrm{Ind}_{G_0}^{G}(\rho)(g^{-1})f\big)(x) = c_\rho(g,x)^{-1} f(g \cdot x)$$

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  and is completely determined as soon as we prescribe the cocycle $c_\rho : G \times X \to \mathrm{GL}(V)$.
- For this, we assume that there is a cross-section $s : G/G_0 \to G$. We set

$$c_\rho(g,x) = \rho(s(g\cdot x)^{-1}gs(x)),\ \ x \in X,\ \ g \in G.$$

# The verification that $c_\rho$ is a cocycle

- We have to verify that $c_\rho$ is a cocycle. First, note that if $g \in G$, then there is a unique $g_x \in G$ such that

$$g = g_x \cdot g_0, \; g_0 \in G_0, \; g_x(o) = x.$$

(To find $g_0$, observe that $g_x^{-1}(g \cdot o) = g_x^{-1}(x) = o$. Therefore, $g_x^{-1} g$ is in $G_0$, that is, $g_x^{-1} g = g_0$ for some $g_0 \in G_0$. Consequently, $g = g_x g_0$.)

Thus, we may pick $s$ to be the cross-section: $s(x) = g_x$. Moreover, every $g \in G$ corresponds to a $x \in X$:

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- It turns out that $s(g \cdot x)^{-1} g s(x)$ is in $G_0$. To verify this, we check its action on $o$:

$$\big(s(g \cdot x)^{-1} g s(x)\big)(o) = s(g \cdot x)^{-1} g g_x \cdot o = g_{g \cdot x}^{-1} \cdot (g \cdot x) = o.$$

Next, we have to verify that $c_\rho$ is a cocycle.

# Enough to check $c$ is a cocycle!

- Since $\rho$ is a homomorphism, it follows that $c_\rho$ is a cocycle if $c$ is a cocycle. To check this, it is enough to prove that $c$ is a cocycle:

$$\begin{aligned} c(g_1, g_2x)c(g_2, x) &= s(g_1 \cdot (g_2 \cdot x))^{-1} g_1 s(g_2 \cdot x) s(g_2 \cdot x)^{-1} g_2 s(x) \\ &= s(g_1 g_2 \cdot x)^{-1} g_1 g_2 s(x) \\ &= c(g_1 g_2, x). \end{aligned}$$

  Therefore, we conclude that $\mathrm{Ind}_{G_0}^{G}(\rho)$ is a homomorphism from $G$ into $\mathrm{InvLin}(\mathcal{F}, V)$.

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  Therefore, we conclude that $\mathrm{Ind}_{G_0}^{G}(\rho)$ is a homomorphism from $G$ into $\mathrm{InvLin}(\mathcal{F}, V)$.

- A more common description of the induced representations is the regular representation on the space right invariant functions defined on the group $G$. We discuss this next.

# Regular representations

- Let $(\rho, V)$ be a representation of the stabilizer group $G_0$. Let

$$\mathcal{F}(G,V)^{\rho} := \left\{F \in \mathcal{F}(G,V) \mid F(gh) = \rho(h)^{-1}F(g),\ h \in G_0\right\}.$$

Now, the $G$ action on $\mathcal{F}(G,V)^{\rho}$ by left translation, namely,

$$\left(L_g^{\rho}F\right)(\hat{g}) = F(g^{-1}\hat{g}), \hat{g} \in G,$$

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- If $\rho \sim \rho'$, that is, $\rho'(h) = A\rho(h)A^{-1}$ for some $A \in \mathrm{GL}(V)$, then $L^\rho \sim L^{\rho'}$ via the map $F \to F' := AF$.

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**Lemma**

*For any multiplier $c$ (writing $\rho_c(h) = c(h, o)$) the map $f \mapsto \tilde{f}$ defined by*

$$\tilde{f}(g) = c(g, o)^{-1} f(g \cdot o), f \in \mathcal{F}(X, V),$$

*maps $\mathcal{F}(X, V)$ to $\mathcal{F}(G, V)^\rho$ and intertwines $T^c$ with $L^\rho$.*

# Equivalence of the regular and multiplier representations

Given a representation $(\rho, V)$ and a global cross-section $s$, i.e. if $g = g_x g_0$, then $s(x) = g_x$, define $\hat{s} : \mathcal{F}(G, V)^\rho \to \mathcal{F}(X, V)$ by $\hat{s}(F) = F \circ s$.

**Theorem**

*Given a representation $(\rho, V)$, the induced map $\hat{s}$ intertwines $L^\rho$ and $T^{c_\rho}$. Moreover, the inverse $\hat{s}^{-1}$ is given by the formula $\hat{s}^{-1}(f) = \rho(s(o))\tilde{f}$, where $\tilde{f}(g) = c(g, o)^{-1} f(g \cdot o)$, $f \in \mathcal{F}(X, V)$.*

The verification that $\hat{s}$ intertwines $L^\rho$ and $T^{c_\rho}$ is straightforward. Also, the string of equalities below

$$\begin{aligned}
\hat{s}^{-1}\big(\hat{s}(F)\big)(g) &= \rho(s(o))\big(\widetilde{F \circ s}\big)(g)\big) \\
&= \rho(s(o)) c(g, o)^{-1} (F \circ s)(g \cdot o) \\
&= \rho\big((s(g \cdot o)^{-1} g s(o))(s(o))^{-1}\big)^{-1} F(s(g \cdot o)) \\
&= F(s(g \cdot o) s(g \cdot o)^{-1} g s(o)(s(o))^{-1}) = F(g)
\end{aligned}$$

proves that the inverse of $\hat{s}$ is given by the formula $f \mapsto \rho(s(o))\tilde{f}$, $\tilde{f}(g) = c(g, o)^{-1} f(g \cdot o)$.

# Thank You!