The Grothendieck inequality

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max Cut

- A cut in a undirected graph $G = (V, E)$ is defined as partition of the vertices of G into two sets; and the weight of a cut is the number of edges that has an end point in each set, that is, the edges that connect vertices of one set to the vertices of the other.
- The max-cut is the problem of finding a cut in G with maximum weight.
- As an example, we note that the bipartite graph has maxcut exactly equal to the number of its edges.
- This is the MAX-2COLORING problem, namely, that of finding the maximum number of edges in a graph G which can be colored by using only two colors.

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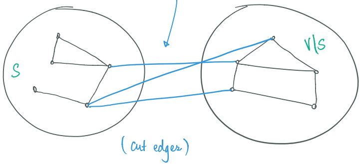
the edge set

Max Cut.

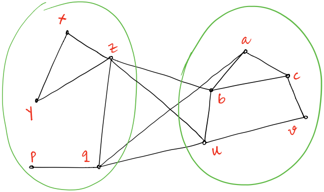
A cut in a graph $G = (V, E)$ is a pair $(S, V \setminus S)$.

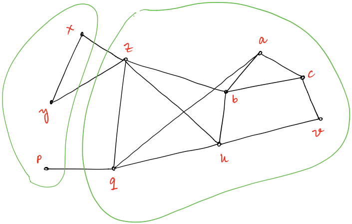
The edge set of the cut is the set of all edges

$$E(S, V\setminus S) = \{ e \in E \mid |e \cap S| = |e \cap V\setminus S| = 1 \}$$

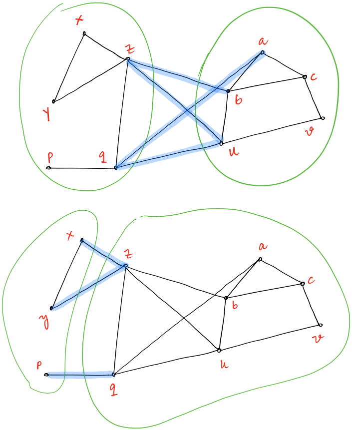


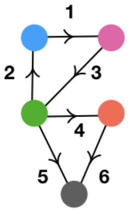
the edge set with labels





the edge set with crossings marked

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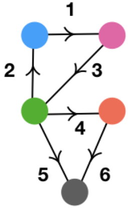
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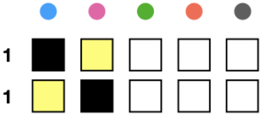
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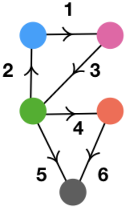
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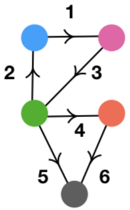
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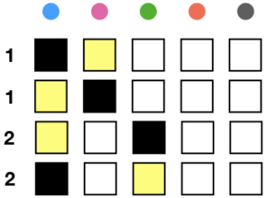
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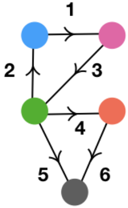
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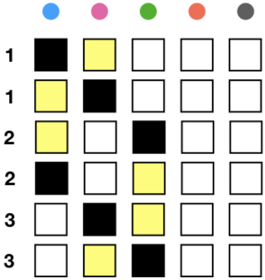
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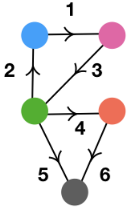
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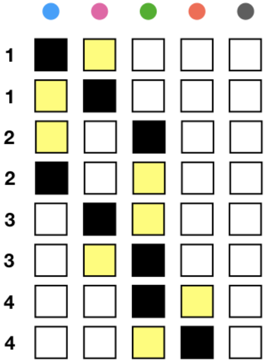
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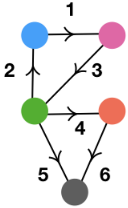
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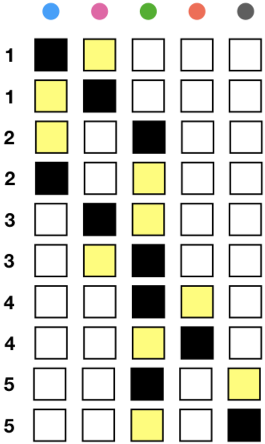
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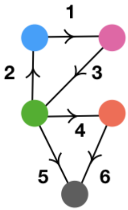
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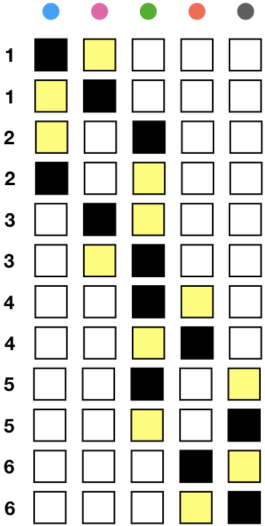
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<u>Cut Norm</u>

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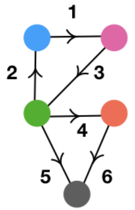
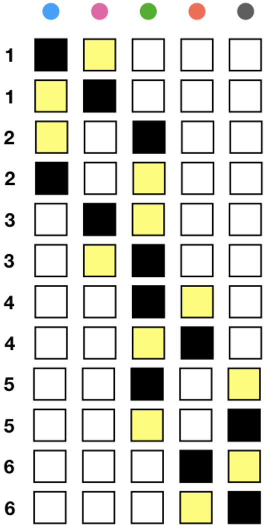
$$\left|\sum_{i \in I, j \in J} a_{ij}\right|$$





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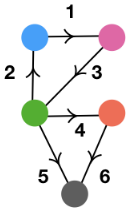

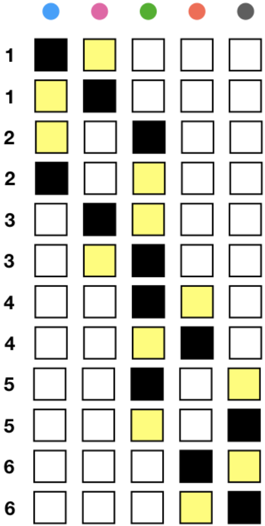

Claim: The cut norm
(of the matrix on the right)
is equal
to the size of the max cut
(of the graph on the left).

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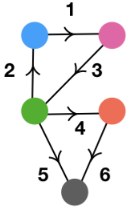

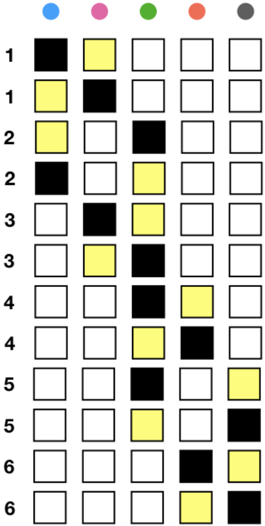

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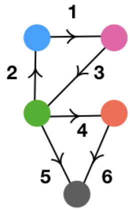
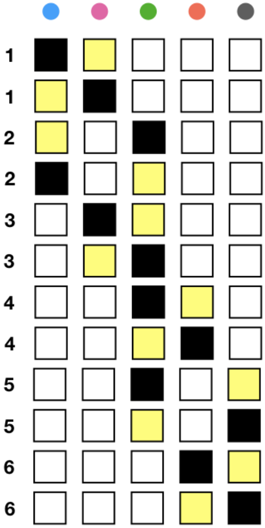


Claim: The cut norm is at least the size of the max cut.



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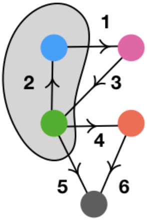

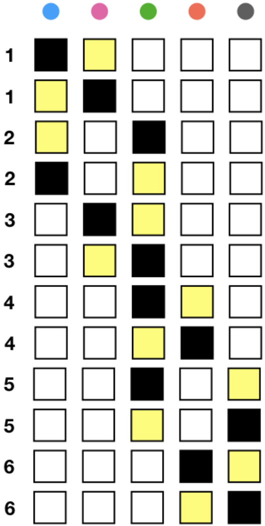

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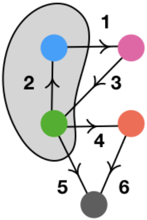

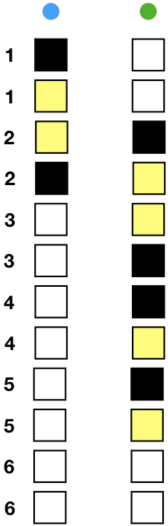

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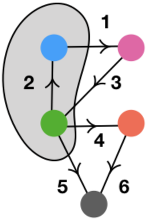

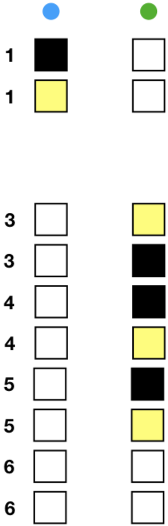

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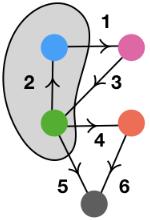

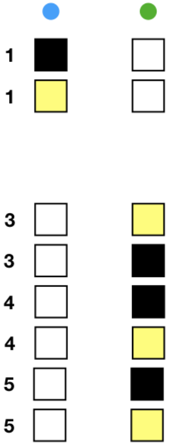

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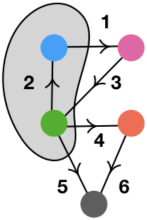

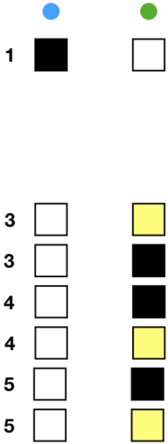

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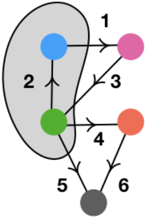

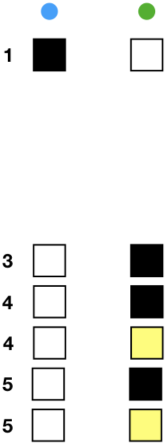

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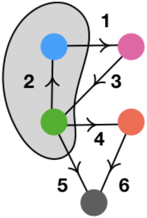

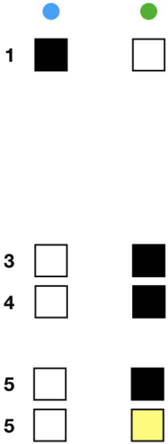

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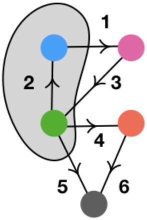

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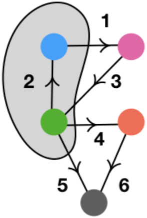

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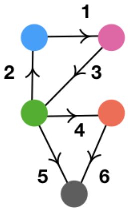

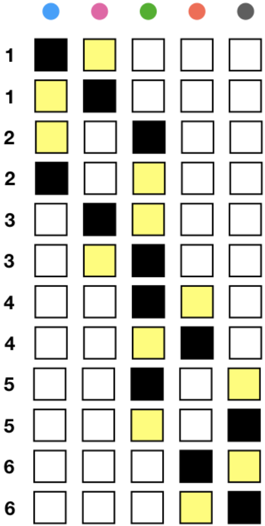

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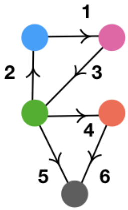

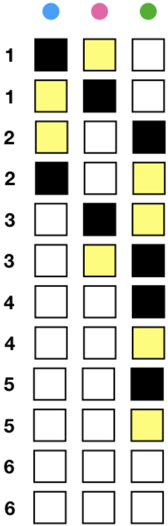

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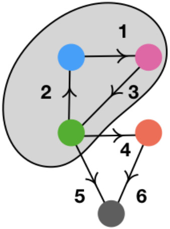

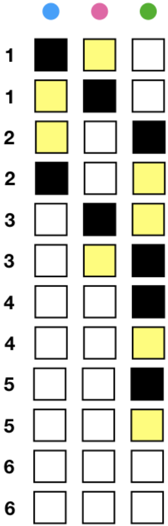

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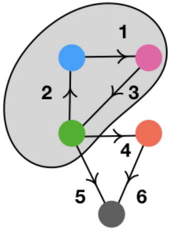

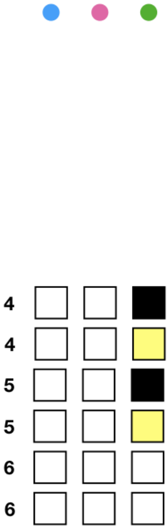

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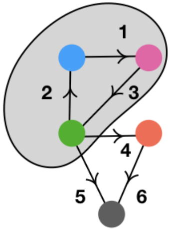

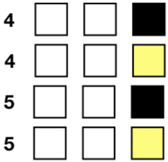

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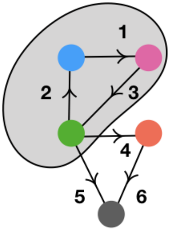

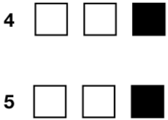

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the Cut norm

• The cut-norm $\|A\|_C$ of a real matrix $A = \Big(\big(a_{ij}\big)\Big)_{i\in R, j\in S}$ is the maximum, over all $I \subseteq R, J \subseteq S$, of the quantity $\left|\sum_{i\in I, j\in J} a_{ij}\right|$.
• It is not difficult to show that the norm $\|\cdot\|_C$ is equivalent to the norm $\|A\|_{\infty\to 1}$, that is, for any $n \times n$ matrix A, we have

$$4\|A\|_C \geq \|A\|_{\infty\to 1} \geq \|A\|_C,$$

where

$$\|A\|_{\infty\to 1} := sup\left\{\left|\sum_{j,k=1}^{n} a_{jk} s_j t_k\right| : |s_j|, |t_k| = 1, 1 \leq j, k \leq n\right\},$$

$s_j, t_k \in \mathbb{R}$ (resp. in $\mathbb{C}$).

proof

For any $x_i, y_j \in \{-1, 1\}$,

$$\sum_{i,j} a_{i,j} x_i y_j = \sum_{i:x_i=1, j:y_j=1} a_{i,j} - \sum_{i:x_i=1, j:y_j=-1} a_{i,j}$$

$$- \sum_{i:x_i=-1, j:y_j=1} a_{i,j} + \sum_{i:x_i=-1, j:y_j=-1} a_{i,j}.$$

The absolute value of each of the four terms in the right hand side is at most $\|A\|_C$, implying, by the triangle inequality, that

$$\|A\|_{\infty \to 1} \leq 4\|A\|_C.$$

proof (contd.)

Suppose, now, that $\|A\|_C = \sum_{i \in I, j \in J} a_{i,j}$ (the computation in case it is $-\sum_{i \in I, j \in J} a_{i,j}$ is essentially the same). Define $x_i = 1$ for $i \in I$ and $x_i = -1$ otherwise, and similarly, $y_j = 1$ if $j \in J$ and $y_j = -1$ otherwise. Then

$$\|A\|_C = \sum_{i,j} a_{i,j} \tfrac{1+x_i}{2} \tfrac{1+y_j}{2} =$$

$$\tfrac{1}{4}\left(\sum_{i,j} a_{i,j} + \sum_{i,j} a_{i,j} x_i \cdot 1 + \sum_{i,j} a_{i,j} 1 \cdot y_j + \sum_{i,j} a_{i,j} x_i y_j \right).$$

The absolute value of each of the four terms in the right hand side is at most $\|A\|_{\infty \to 1}/4$, implying, by the triangle inequality, that

$$\|A\|_{\infty \to 1} \geq \|A\|_C.$$

integer linear program

- Finding the norm $\|A\|_{\infty\to1}$ is called an integer linear program since

$$\|A\|_{\infty\to1} := sup\left\{\left|\sum_{j,k=1}^{n} a_{jk}s_jt_k\right| : s_j, t_k \in \{-1,1\}, 1 \le j,k \le n\right\},$$

at least in the real case.

- Thus one may wish to simply compute the $\|A\|_{\infty\to1}$ instead of the CUT norm. However, this is not easy either.
- Let us see if we can give ourselves a little more room and compute a norm, namely, the 2-summing norm, related to the cut norm and the norm $\|A\|_{\infty\to1}$ that we have already seen.

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the LP relaxation

- The 2 - summing norm $\gamma(A)$ is defined as follows:

$$\gamma(A) := \sup\left\{\Big|\sum_{j,k=1}^{n} a_{jk}\langle x_j, y_k\rangle\Big| : x_j, y_k \in \left(\ell_2\right)_1, 1 \le j,k \le n\right\}.$$

Finding $\gamma(A)$, the 2 - summing norm, is called a semi-definite program.

- Define the numerical constant, the Grothendieck constant:

$$K_G(n) \stackrel{def}{=} \sup\{\gamma(A) : A = A_{n\times n}, \|A\|_{\infty\to 1} \le 1\}.$$

- The constant $K_G(n)$ depends on the ground field.

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what we know about the Grothendieck constant

• The fact that $K_G(n)$ remains finite, say K_G, as $n \to \infty$ was established by Grothendieck and is known as the Grothendieck constant, that is,

$$\sup\{\frac{\gamma(A)}{\|A\|_{\infty\to 1}} : A \in \mathbb{C}^{n\times n},\ n \in \mathbb{N}\} < \infty.$$

• The Grothendieck inequality says that the two norms $\|A\|_{\infty\to 1}$ and $\gamma(A)$ can differ only by a constant factor.
• The exact value of K_G is not known. However, $K_G^{\mathbb{C}}(1) = K_G^{\mathbb{C}}(2) = 1$ and $K_G^{\mathbb{R}}(2) = \sqrt{2} = K_G^{\mathbb{R}}(3)$.
• Although, not entirely trivial, it is known that $K_G > 1$.
• Kirvine's proof gives $\frac{\pi}{2\ln(1+\sqrt{2})} = 1.782\ldots$.
• Krivine conjectured that his bound is actually the exact value of K_G. Recently, this conjecture has been shown to be false.

Grothendieck constant for graphs

• Let G be a graph with n vertices denoted by $\{1, \dots, n\}$ and $E \subseteq \{1, \dots, n\}^2$ be the set of its edges.
• Following Noga Alon, Assaf Naor and many others, define the Grothendieck constant of the graph G, denoted by $K(G)$, to be the smallest constant K such that

$$\sup\Big\{|\sum_{\{i,j\}\in E} a_{ij}\langle xi, y_j\rangle| : \|x_i\| = 1 = \|y_j\|\Big\} \leq$$

$$K \sup\Big\{\sum_{\{i,j\}\in E} a_{ij} s_i t_j : |s_i| = 1 = |t_j|\Big\}$$

holds true for any real matrix $A = \Big(\Big(a_{ij}\Big)\Big)$.

the original Grothendieck inequality

• The original Grothendieck inequality is the particular case that corresponds to the bipartite graphs (i.e. of chromatic number 2) and, as a consequence,

$$K_G = \sup_{n\in\mathbb{N}} \Big\{ K(G) : \quad G \text{ is a bipartite graph on } n \text{ vertices} \Big\}.$$

• Additionally, if C_n stands for the complete graph with n vertices, the corresponding Grothendieck constant is of order $\log(n)$. The Grothendieck constant of a graph G is clearly related to the combinatorics of G.

???

• On the other hand, the expression on the right hand side of the Grothendieck inequality for graphs is relevant statistical physics: if G weighted by the matrix A represents the possible interaction of n particles affected by a spin $i = \pm 1$, then the total energy generated by these particles in the system in the Ising model of the spin glass is

$$\mathcal{E} = -\Big(\sum_{\{i,j\} \in E} a_{ij} \varepsilon_i \varepsilon_j \Big).$$

A configuration of the spins $(\varepsilon_i) \in \{-1, 1\}^n$ represents its ground state if it minimizes the energy.

Kirvine's proof of the Grothendieck inequality

Let $S \subseteq \mathbb{C}^k$ be the Euclidean sphere of radius 1.

Lemma

$$\sup\left\{\left|\sum_{i,j=1}^n a_{ij}\sin^{-1}\langle u_i, v_j\rangle\right| : \|A\|_{\infty\to 1} \le 1; u_i, v_j \in S\right\} \le \frac{\pi}{2}.$$

Proof. Let μ be the unique probability measure on S which is rotation invariant. First, show that

$$I := \int_S sign\langle x, u\rangle sign\langle y, u\rangle d\mu(u) = 1 - \frac{2\psi}{\pi}, \psi = \cos^{-1}\langle x, y\rangle, x, y \in S.$$

- The verification consists of finding an unitary $U : \ell_2(k) \to \ell_2(k)$ with

$$Ux = (1, 0, \ldots, 0),\ Uy = (\cos\psi, \sin\psi, 0, \ldots, 0),$$

where $\psi = \cos^{-1}\langle x, y\rangle$, $0 \le \psi \le \pi$ and $\sin^{-1}\langle x, y\rangle = \frac{\pi}{2} - \psi$.

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Kirvine's proof

- If x and y are linearly dependent, namely $x = -y$, then $Ux = (1, 0, \ldots, 0)$, $Uy = (-1, 0, \ldots, 0)$ and $\psi = \pi$. Similarly, if $x = y$, then choose $Ux = (1, 0, \ldots, 0)$, $Uy = (1, 0, \ldots, 0)$ and $\psi = 0$. Now, extend this map linearly to all of $\ell_2(k)$ to an unitary.

- If x and y be linearly independent, then applying Gram-Schimdt, obtain a pair of orthonormal vectors α_1, α_2 and define a linear map U on the span of these two vectors:

$$U\alpha_1 := (1, 0, \ldots, 0),\ U\alpha_2 := (0, 1, 0, \ldots, 0)$$

and extend it, as before, to an unitary on all of $\ell_2(k)$.

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an integral

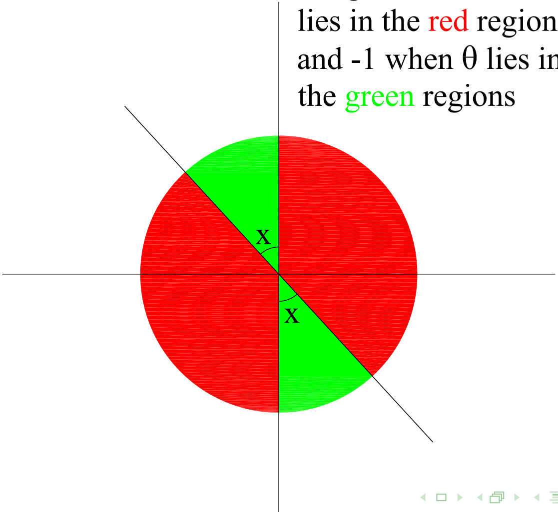
- A simple calculation gives $Ux = (1, 0, \ldots, 0)$, $Uy = (\cos\psi, \sin\psi, 0, \ldots, 0)$. Therefore, in computing $\langle Ux, Uu\rangle$ and $\langle Uy, Uu\rangle$, we assume without loss of generality: $Uu = (\cos\theta, \sin\theta, 0 \ldots, 0)$.
- The integral I is U invariant, we have

$$\begin{aligned}
I &= \int_S sign\langle Ux, Uu\rangle sign\langle Uy, Uu\rangle d\mu(Uu) \\
&= \int_S sign\, u_1 sign(\cos\psi u_1 + \sin\psi u_2) d\mu(Uu) \\
&= \frac{1}{2\pi}\int_0^{2\pi} sign(\cos\theta) sign(\cos(\theta - \psi)) d\theta \\
&= 1 - \frac{2\psi}{\pi} \\
&= \frac{2}{\pi}\sin^{-1}\langle x, y\rangle.
\end{aligned}$$

evaluation of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} sign(\cos\theta) sign(\cos(\theta - x))d\theta$$

Integrand is +1 when θ lies in the red regions and -1 when θ lies in the green regions



tensor product

- The hypothesis on A implies that

$$-1 \leq \sum_{i,j=1}^{n} a_{ij} sign\langle u_i, x\rangle sign\langle v_j, x\rangle \leq 1,$$

for any choice of vectors $\|u_i\|_2 = 1 = \|v_j\|_2$. The proof is then completed by integrating with respect to x. $\square$

Lemma

For each positive integer k, there is a mapping $w_k : l_2^n \to l_2^N$ such that for all x, y, $\langle w_k(x), w_k(y)\rangle = \langle x, y\rangle^k$.

- For the proof, set $w_k(x)$ to be the k - fold tensor product of the vector x.

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sine hyperbolic

Lemma

Given $c > 0$, there exists $u : \ell_2(n) \to \ell_2$ and $v : \ell_2(n) \to \ell_2$ such that

$$\langle u(x), v(y)\rangle = \sin c\langle x, y\rangle,$$

$\|u(x)\|^2 = \sinh\left(c\|x\|^2\right)$ *and* $\|v(y)\|^2 = \sinh\left(c\|y\|^2\right)$, $x, y \in \ell_2(n)$.

Proof. From the Taylor series expansion

$$\sin c\langle x, y\rangle = \sum_1^\infty (-1)^{k-1} c_k \langle w_{2k-1}(x), w_{2k-1}(y)\rangle,$$

where $c_k = \frac{c^{2k-1}}{(2k-1)!}$, we see that we just have to set

$$\begin{aligned} u(x) &:= \sum_1^\infty \sqrt{c_k}\, w_{2k-1}(x), \\ v(y) &:= \sum_1^\infty (-1)^{k-1} \sqrt{c}_k\, w_{2k-1}(y). \end{aligned}$$

completing the proof

- Let $c = \sinh^{-1}(1) = \ln(1+\sqrt{2})$.
  Set $u_i = u(x_i)$, $v_j = v(y_j)$, $\|x_i\|_2 = 1 = \|y_j\|_2$, and note that $\|u_i\| = 1 = \|v_j\|$.
- However, we know that

$$c\langle x_i, y_j\rangle = \sin^{-1}\langle u_i, v_j\rangle, \mid c\langle x_i, y_j\rangle \mid \leq 1$$

and

$$\mid \sum_{i,j=1}^{n} a_{ij} \sin^{-1}\langle u_i, v_j\rangle \mid \leq \frac{\pi}{2}.$$

So

$$\mid \sum_{i,j=1}^{n} a_{ij}\langle x_i, y_j\rangle \mid \leq \frac{\pi}{2c} = \frac{\pi}{2\ln(1+\sqrt{2})}.$$

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Theorem (Varopoulos inequality)

Suppose $K_G^{\mathbb{C}}$ *denote the complex Grothendieck constant. Then*

$$K_G^{\mathbb{C}} \leq \sup \|p(T_1, \dots, T_n)\| \leq 2K_G^{\mathbb{C}}$$

where supremum is over all $n \in \mathbb{N}$, *tuples of commuting contractions* $T = (T_1, \dots, T_n)$ *and polynomial* p *of degree 2 with* $\|p\|_\infty \leq 1$.

sharpening the Varopolous inequality

- Thus Grothendieck constant had made an unexpected appearance in the early work of Varopoulos. Setting

$$C_2(n) = \sup\Big\{\|p(T)\| : \|p\|_{\mathbb{D}^n,\infty} \leq 1, \|T\|_\infty \leq 1\Big\},$$

where the supremum is taken over all complex polynomials p in n variables of degree at most 2 and commuting n - tuples $T := (T_1, \ldots, T_n)$ of contractions, he shows that

$$\lim_{n\to\infty} C_2(n) \leq 2K_G^{\mathbb{C}},$$

where $K_G^{\mathbb{C}}$ is the complex Grothendieck constant.

- Rajeev Gupta in his PhD thesis shows that

$$\lim_{n\to\infty} C_2(n) \leq \frac{3\sqrt{3}}{4}K_G^{\mathbb{C}},$$

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