Simultaneous invariant subspaces of $L^2(\mathbb{D})$

Joint Spectra and related Topics in Complex Dynamics
and Representation Theory
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A question of invariant subspaces

group action, homogeneous operators, associated representation

definitions and results

What are the associated representations

invariant subspace

main result

A question of invariant subspaces

a question of invariant subspaces

- For a fixed $\lambda \in \mathbb{R}$, let L^λ be the Hilbert space of measurable (complex-valued) functions on the open unit disc $\mathbb{D}$ such that

$$\|f\|_\lambda^2 = \int_{\mathbb{D}} (1-|z|^2)^{\lambda-2} |f(z)|^2 dA_z < \infty.$$

(dA_z stands for $dxdy$, $z = x + iy$.)

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- Let G be the simply connected covering group of $SU(1,1); g(z)$ and $g'(z) = \frac{\partial(g(z))}{\partial z}$ still make sense for $g \in G$. The group G acts on L^λ by a unitary representation $\{U_g\}$ via the multiplier $g'(z)^{-\lambda/2}$.

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- Explicitly (it is simpler to write down $U_{g^{-1}}$ than U_g),

$$U_{g^{-1}} f(z) = g'(z)^{\lambda/2} f(g(z)).$$

  Unitarity follows from the identity $(1-|z|^2)|g'(z)| = 1-|g(z)|^2$.

(U_g is the representation induced in Mackey's sense by the character $-\frac{\lambda}{2}$ of K, the stabiilizer of 0 in G. This follows by observing that any $g \in K$ acts by a rotation $g(z) = e^{i\theta} z$, and then $g'(z)^{-\lambda/2} = e^{-i\frac{\lambda}{2}\theta}$.)

an invariant subspace

- The subspace H^λ of holomorphic functions in L^λ is invariant under $\{U_g\}$.
  ($H^\lambda \neq 0$ if and only if $\lambda > 1$; in this case, the restriction of $\{U_g\}$ to H^λ is the holomorphic discrete series.)

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- H^λ is also an invariant subspace for the operator M defined by $Mf(z) = zf(z)$.
- Question: Is H^λ the only subspace invariant under both $\{U_g\}$ and M?
  Note: A similar question can be asked if we regard $\{U_g\}$ as a representation on $C^\infty(\mathbb{D})$.

Mackey’s theorem, multiplier and induced representations

imprimitivity

- Let G be a connected Lie group, $H \subseteq G$ closed, $\mathcal{D} = G/H$, and $0 \in \mathcal{D}$ be the base point. The group G acts transitively, via $L_g(g'H) = gg'H$, on the space of cosets G/H.

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- suppose that $U : G \to \mathcal{U}(\mathcal{K})$ is a unitary representation of the group G on the Hilbert space $\mathcal{K}$ and that $\varrho : \boldsymbol{C}_0(\mathcal{D}) \to \mathcal{L}(\mathcal{K})$ is a $*$ - homomorphism of the C^* - algebra of continuous functions $\boldsymbol{C}_0(\mathcal{D})$ vanishing at ∞ on the algebra $\mathcal{L}(\mathcal{K})$ of all bounded operators acting on the Hilbert space $\mathcal{K}$.

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- Then setting $(g \cdot f)(w) = f(g^{-1} \cdot w), w \in \mathcal{D}$, the pair (U, ϱ) is said to be a representation of the G -space $\mathcal{D}$ if it forms an imprimitivity:

$$U(g)^* \varrho(f) U(g) = \varrho(g \cdot f) =, f \in \boldsymbol{C}_0(\mathcal{D}), g \in G.$$

multiplier representation

- A multiplier is a C^∞ map $m: G \times \mathcal{D} \to \mathsf{GL}(V)$ such that

$$m(gh,z) = m(g,hz)m(h,z),\ g,h \in G,\ z \in \mathcal{D}.$$

It follows that $m(e,z) \equiv I_V$ and that m is determined from the values $m(g,0)$, $g \in G$, and that $\sigma_m(h) := m(h,0)$ is a representation of H. Finally, $m(g,g^{-1}z) = m(g^{-1},z)^{-1}$, $g \in G, z \in \mathcal{D}$. The multiplier representation U^m acts on $C^\infty(\mathcal{D},V)$:

$$(U^m_g f)(z) = m(g^{-1},z)^{-1} f(g^{-1}z),\ f \in C^\infty(\mathcal{D},V),\ z \in \mathcal{D}.$$

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- Conversely, the map $\pi: G\to C^\infty(\mathcal{D},V)$, called a multiplier representation,

$$(\pi(g)f)(z)=(m(g^{-1},z))^{-1}f\left(g^{-1}z\right),\ z\in\mathcal{D}, f\in\mathcal{H}, g\in G,$$

  is a homomorphism if and only if $m: G\times\mathcal{D}\to\mathrm{GL}(V)$ is a multiplier.

induced representation

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- There is a measurable cross section (actually can be C^∞ or holomorphic depending on the pair of groups G, H) $s: G/H \to G$, i.e., $p \circ s = \mathrm{id}_{|G/H}$.
- There is a unique quasi-invariant measure μ on G/H , i.e., $\mu \circ L_g$ and μ are absolutely continuous for all $g \in G$.

Mackey's imprimitivity theorem

- Set $\mathcal{K} := L^2(G/H, \mu, V)$. The representation Ind induced from a unitary representation σ of H acting on V is then given by the formula

$$(\mathrm{Ind}(g)f)(g'H) = \frac{d\left(\mu \circ L_g^{-1}\right)}{d\mu}\sigma(h)(f \circ L_g^{-1})(g'H), g \in G,$$

where h is determined from the relation $gs(L_g^{-1}g'H) = s(g'H)h.$

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- Mackey's Imprimitivity theorem says that any pair (U, ϱ) of imprimitivity is necessarily an induced representation.
- Mackey described these modulo suitable equivalence.
- Question: If we replace the $*$ -homomorphism ρ of a commutative C^* -algebra by an algebra homomorphism, then what should we expect? Answer: These are the commuting tuples of homogeneous operators!

group action, homogeneous operators, associated representation

the Möbius group, the 2 - fold covering group $\mathrm{SU}(1,1)$

- The Möbius group, to be denoted G_0 in what follows, consists of the bi-holomorphic automorphisms of $\mathbb{D}$ of the form $\varphi_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$, $0 \leq \theta < 2\pi$, and a in the unit disc $\mathbb{D}$.

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- The linear group $G := \mathrm{SU}(1,1)$ is the group of complex 2×2 matrices of the form $g := \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 - |\beta|^2 = 1$. It acts on the unit disc by the rule $g(z) = \frac{\alpha z+\beta}{\bar{\alpha}+\bar{\beta}z}$.

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- Clearly, the element g and $-g$ give rise to the same action. One may say that the kernel of the $\mathrm{SU}(1,1)$ action on the unit disc is the normal subgroup $\{I, -I\}$.

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- We let $\tilde{G}$ denote the universal covering group of G.

homogeneous operators

- The biholomorphic map $\varphi_{\theta,a}$ comes from the action of a $g \in \mathrm{SU}(1,1)$, namely,

$$g = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} \exp(i\frac{\theta}{2}) & 0 \\ 0 & \exp(-i\frac{\theta}{2}) \end{pmatrix} \left(\begin{smallmatrix} 1 & -a \\ -\bar{a} & 1 \end{smallmatrix}\right),$$

and also from $g_{\theta+2\pi,a}$.

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- A bounded linear operator T on a complex separable Hilbert space $\mathcal{H}$ with $\sigma(T) \subseteq \bar{\mathbb{D}}$ is said to be homogeneous if $\varphi_{\theta,a}(T) := e^{i\theta}(T - aI)(I - \bar{a}T)^{-1}$ is unitarily equivalent to T for all $\varphi_{\theta,a} \in G_0$.

definitions and results

homogeneous operators and associated representations

- The $*$ - homomorphism ρ_N induced by N, namely, $\rho_N(f) = f(N)$ satisfying the imprimitivity condition is easily checked to be the condition of homogeneity for the operator N.

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- To every homogeneous irreducible operator T there corresponds an associated unitary representation $\tilde{\pi}$ of the universal covering group $\tilde{G}$:

$$\tilde{\pi}(\tilde{g})^* T \tilde{\pi}(\tilde{g}) = (p\tilde{g})(T),\ \tilde{g} \in \tilde{G},$$

  where $p : \tilde{G} \to G$ is the natural homomorphism.

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- If T is contractive irreducible homogeneous operator, then the homomorphism ρ_T induced by T, namely, $\rho_T(p) = p(T)$ is the compression of an imprimitivity.

subnormal operators

- An operator T on a Hilbert space $\mathcal{H}$ is said to be subnormal if there exists a normal operator N on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is an invariant subspace for N and $N_{|\mathcal{H}} = T$.

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- Two such normal extensions are unitarily equivalent if they are assumed to be minimal, that is, $\mathcal{K}$ is the smallest reducing subspace of N containing $\mathcal{H}$.

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- A minimal normal extension, say mne(T), always exists and is uniquely determined modulo unitary equivalence.

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- Two such normal extensions are unitarily equivalent if they are assumed to be minimal, that is, $\mathcal{K}$ is the smallest reducing subspace of N containing $\mathcal{H}$.
- A minimal normal extension, say mne(T), always exists and is uniquely determined modulo unitary equivalence.
- If T is a subnormal irreducible homogeneous operator, then the homomorphism ρ_T is the restriction to some simultaneous invariant subspace of an imprimitivity.

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- Moreover, if a subnormal homogeneous operator has an associated representation, say U, then together with T, the representation U also extends to an associated representation of the mne(T).

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- Moreover, if a subnormal homogeneous operator has an associated representation, say U, then together with T, the representation U also extends to an associated representation of the mne(T).
- A list of all the normal operators that are homogeneous is not hard to produce. For these, we also determine all the associated representations. The result suggests a possible approach to finding all the homogeneous subnormal operators:

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- In other words, if T is subnormal and homogeneous, then the mne(T) is also homogeneous.
- Moreover, if a subnormal homogeneous operator has an associated representation, say $U,$ then together with $T,$ the representation U also extends to an associated representation of the mne(T).
- A list of all the normal operators that are homogeneous is not hard to produce. For these, we also determine all the associated representations. The result suggests a possible approach to finding all the homogeneous subnormal operators:
- Every homogeneous subnormal operator is the restriction to a common invariant subspace of a homogeneous normal operator and one of its associated representations.

Theorem

Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator. Then N is homogeneous if and only if there exists $m, m' \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ such that $N = A_m \oplus B_{m'}$, where A_m is the m -fold direct sum of M_z on $L^2(\mathbb{D}, dA)$ and $B_{m'}$ is m' -fold direct sum of M_z on $L^2(\mathbb{T}, d\theta)$.

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- The question of determining which homogeneous operators are subnormal then is equivalent to asking what are the subspaces of

$$(\oplus_m L^2(\mathbb{D}, dA)) \oplus (\oplus_{m'} L^2(\mathbb{T}, d\theta))$$

simultaneously invariant under the unitary representation U and the multiplication operator M.

tools and techniques

- All homogeneous operators in $B_m(\mathbb{D})$, modulo unitary equivalence, are known. We identify which operators among these are subnormal, that is, appear as the restriction of a normal operator to an invariant subspace. For this, we need some tools from representation theory which we recall. Let X be either $\mathbb{D}$ or $\mathbb{T}$.

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- Let V be a vector space and for each fixed $g \in \tilde{G}$, $m : \tilde{G} \times X \to \mathrm{GL}(V)$ be a Borel multiplier: $m(e,z) = I_V, \;\; m(g_1 g_2, z) = m(g_1, g_2 z) m(g_2, z), z \in X,$

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- Let $\mathcal{H}$ be a Hilbert space of V -valued functions on Ω. A representation admitting a realization of the form

$$(\pi(g)f)(z) = (m(g^{-1},z))^{-1} f\,(g^{-1}z)\,,\ z \in X, f \in \mathcal{H}, g \in \tilde{G}$$

is said to be a multiplier representation.

What are the associated representations

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Theorem

For M on $L^2_V(d\theta)$, dim $V < \infty$, the associated representations are exactly the orthogonal direct sums (with possible repetition) of arbitrary principal series representations. They act on the (scalar valued) $L^2(d\theta)$ subspaces by the multiplier $\left(\frac{g'(e^{i\theta})}{|g'(e^{i\theta})|}\right)^{-k} |g'(e^{i\theta})|^{-\frac{1}{2}-i\sigma}$, $-\frac{1}{2} < k < \frac{1}{2}, \;\; \sigma > 0.$

conventions

- For a given $d \in \mathbb{N}$, let Λ denote a partition $d = d_0 + \cdots + d_m$ together with real numbers $\lambda_0 < \lambda_1 < \ldots < \lambda_m$. Write $d = |\Lambda|$.

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- For $\lambda \in \mathbb{R}$, let χ_λ be the unitary representation of $\mathbb{K}$ on $\mathbb{C}$ given by $\chi_\lambda(k_\theta) = e^{-\lambda\theta}$, where $k_\theta \in \mathbb{K}$ acts in $\mathbb{D}$ by $z \mapsto e^{i\theta} z$. We write

$$\chi_\Lambda(k) = \oplus_j d_j \chi_{\lambda_j}(k),\ k \in \mathbb{K}.$$

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- Modulo unitary equivalence, all the d - dimensional representations of $\mathbb{K}$ are obtained this way with $|\Lambda| = d$. Define, for $\lambda \in \mathbb{R}$,

$$m^\lambda(g, z) = \left(\frac{g'(z)}{|g'(z)|}\right)^{-\lambda}, \quad m^\Lambda = \oplus_j d_j m^{\lambda_j}.$$

Theorem
For M acting on $L^2_V(dA)$, the associate representations are exactly the unitary multiplier representations given by m^Λ with $d = |\Lambda|$.

- The proof involves the imprimitivity theorem due to Mackey establishing the equivalence of the associated representation with the induced representation $\text{Ind}_K^G(\chi^\Lambda)$.

Theorem
For M acting on $L^2_V(dA)$, the associate representations are exactly the unitary multiplier representations given by m^Λ with $d = |\Lambda|$.

- The proof involves the imprimitivity theorem due to Mackey establishing the equivalence of the associated representation with the induced representation $\mathrm{Ind}_K^G(\chi^\Lambda)$.

- Moreover the associated representation of the direct sum $M \oplus M$ acting on $L^2_V(d\theta) \oplus L^2_{V'}(dA)$ is necessarily the direct sum of the ones we have already found.

invariant subspace

invariant subspaces

Theorem
For dim $V < \infty$, the U - invariant subspaces of $L^2_V(d\theta)$ are the orthogonal direct sums of 1 - dimensional subspaces with U acting on them as

$$\oplus_{(k,s)\neq(0,0)} d_{k,s} P_{ks} \oplus d^+ P^+ \oplus d^- P^-, \quad d_{k,s} \in \mathbb{N}.$$

Let S be the restriction of M acting on $L^2(d\theta)$ to the Hardy space H^2.

Corollary
Any homogeneous subnormal operator whose mne is M on $L^2_V(\theta)$, must be the orthogonal direct sum $U \oplus dS$, where U is a homogeneous unitary and $d \in \mathbb{N}$.

quasi-invariant

- Let Q be a ν -measwable map of $\mathbb{D}$ to the positive definite matirices on V, and let $L^2(V,Q,\nu)$ be the Hilbert space consisting of functions $f:\mathbb{D}\to V$ with norm

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- The operator M on this space is another realization of the "same" normal operator M on $L^2_V(d\nu)$. The unitary isomorphism is given by the map $f \mapsto Q^{1/2}f$.
- A multiplier m on $L^2(V,Q,\nu)$ gives a unitary representation if and only if (easy) Q is quasi-invariant:

$$Q(g\cdot z) = {m(g,z)^*}^{-1} Q(z) m(g,z)^{-1} \quad (\forall g, a.a.z)$$

an important observation

- Given two such spaces, with Q and Q', respectively, m and m', we say m and m' are equivalent if there exists a measurable $\phi : \mathbb{D} \to GL(V)$ such that

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- Cleraly, $f \mapsto f' = \phi f$ is a Hilberet space isomorphism between $L^2_V(Q,\nu)$ and $L^2_V(Q',\nu)$ intertwining the multiplier representations given by m, m' and the multiplication operator M on these two spaces.

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- The case of continuous multiplier is particularly simple, devoid of measure theoretic difficulties. In our situation, every equivalence class of a multiplier contains one that is continuous.

some more observations

- Picking a continuous multiplier m and putting $z = 0$, we can rewrite the quasi-invariance of Q as follows:

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- Also easy to see that $\rho(k) = m(k,0), \quad k \in \mathbb{K},$ is unitary with respect to some inner product $\langle Q^0 \cdot, \cdot \rangle_V$ with some $Q^0 > 0,$ then the map $f \to F$ with $F(g) = m(g,0)^{-1} f(g \cdot 0)$ is an isomorphism onto the L^2_V-space of functions $G \to V$ such thet $F(gk) = \rho(k)^{-1} F(g)$, which is the definition of the induced representations $Ind_K^G(\rho)$.

example

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- The multiplier $g'(z)^{-\lambda}$ acts on $L^2(Qd\nu)$ with $Q(z) = (1-|z|^2)^{2\lambda}$ and preserves the subspace $\mathbb{A}^{(\lambda)}$ of holomorphic functions. This subspace is not $\{0\}$ if and only if $\lambda > 1/2$. When $\lambda > 0$, we have a Hilbert space but the subspace of holomorphic functions is not $\{0\}$ if and only if $\lambda > 1/2$.

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- The restriction of the operator M to the subspace $\mathbb{A}^{(\lambda)}$ is then the subnormal homogeneous operator that we have been looking for.

holomorphic structure

- The operator M on $\mathbb{A}^{(\lambda)}$ can however, also be thought of as the restriction of M on $L^2(\nu)$ to a subspace, namely the space $\mathbb{H}^{(\lambda)}$ of elements of $\mathbb{A}^{(\lambda)}$ multiplied by ϕ^{-1}, i.e, the subspace of $L^2(\nu)$ of function f such that $\phi(z)f(z)$ is holomorphic.

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- This, in turn, can be interpreted as f being holomorphic with respect to a changed complex structure, characterized by $\frac{\partial}{\partial \bar{z}}(\phi(z)^{-1}f(z)) = 0$, computing this amounts to $\frac{\partial}{\partial \bar{z}}f(z) = \frac{\lambda z}{l-|z|^2}f(z)$.

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- This example already gives rise to the difficult open question: If the subspace of holomorphic functions in $L^2((1-|z|^2)^{2\lambda}d\nu)$ is the only subspace simultaneously invariant under M and U?

main result

colorcyan main result

The irreducible homogeneous subnormal operators that can be obtained by restricting $(L_V^2(\nu))$ to the subspace of functions holomorphic with respect to a G-invariant complex structure are parametrized by $\eta > \frac{1}{2}$, Y irreducible, and Q^0 commuting with $\chi_\Lambda(k), k \in \mathbb{K}.$

In particular, all of these are adjoints of operators in the Cowen-Douglas class.

Thank You!