The Arveson-Douglas conjecture for semi-submodules

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(after talking to Soumitra Ghara, Surjit Kumar, Paramita Pramanick, and Kalyan Sinha)

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A `Hilbert module` over the polynomial ring $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \ldots, z_m]$ is a Hilbert space $\mathscr{H}$ which is a $\mathbb{C}[\underline{z}]$ -module if for some $C_p > 0$,

$$\|p \cdot f\| \leq C_p \|f\|, f \in \mathscr{H}, p \in \mathbb{C}[\underline{z}].$$

The multiplication M_j, $1 \leq j \leq m$, by the coordinate functions z_j: $M_j f := z_j \cdot f$, then defines a commutative tuple $\mathbf{M} = (M_1, \ldots, M_m)$ of linear bounded operators acting on $\mathscr{H}$ and vice-versa.

A Hilbert module $\mathscr{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the `Cowen-Douglas class` $\mathrm{B}_n(\Omega)$, $n \in \mathbb{N}$, if

- $\dim \mathscr{H} / \mathfrak{m}_w \mathscr{H} = n < \infty$ for all $w \in \Omega$, where $\mathfrak{m}_w$ is the maximal ideal in $\mathbb{C}[\underline{z}]$ at w and
- there exists a holomorphic choice of linearly independent vectors $\{s_1(w), \ldots, s_n(w)\}$ in $\mathscr{H} / \mathfrak{m}_w \mathscr{H}$.

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essentially r - reductive modules

A Hilbert module $\mathscr{M}$ over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ is said to be essentially r - reductive if the module multiplication $m_p,\ p \in \mathbb{C}[\boldsymbol{z}]$ is essentially normal, that is,

$$[p(\boldsymbol{T})^*, p(\boldsymbol{T})] \in \mathscr{S}_r(\mathscr{M}), \quad p \in \mathbb{C}[\boldsymbol{z}].$$

- The Hardy and the Bergman modules $H^2(\mathbb{D})$ and $\mathbb{A}^2(\mathbb{D})$ over the disc algebra $\mathscr{A}(\mathbb{D})$ are essentially reductive.
- The Drury-Arveson module H^2_m is a module over the polynomial ring $\mathbb{C}[\boldsymbol{z}]$ which is essentially reductive, although, the module multiplication does not extend to the ball algebra.
- Apart from asking if a Hilbert module $\mathscr{M}$ is essentially reductive, one may also ask which submodules and quotient modules are essentially reductive. It is likely to have an interesting answer if $\mathscr{M}$ is essentially reductive.
- Neither the Hardy module nor the Bergman module over the algebra $\mathscr{A}(\mathbb{D}^m), m > 1,$ is essentially reductive. However, some quotient modules of the Hardy module over $\mathscr{A}(\mathbb{D}^m), m > 1,$ are essentially reductive while some are not.

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There is one other possibility that we haven't considered.

- The weighted Bergman modules over a bounded symmetric domain Ω are reductive if and only if the rank of the domain Ω is $1,$ Ghara, Kumar and Pramanick.
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To be specific, take Ω to be the unit ball in $n \times n$ matrices. The Euclidean ball $\mathbb{B}_n$ embeds into it.

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example

- Let $\mathbf{z} \in \mathbb{B}_n$ map to the $n \times n$ matrix whose first column is $\mathbf{z}$ and each of the remaining $(n-1)$ columns is the zero vector of size n.
- If we consider the submodule $\mathscr{S}$ of functions vanishing on $\mathbb{B}_n$ in the weighted Bergman module $\mathscr{H}^{(\nu)}(\Omega)$, then the corresponding quotient module $\mathscr{Q}$ can be identified with the restriction of the functions in $\mathscr{H}^{(\nu)}(\Omega)$ to the zero set, namely, $\mathbb{B}_n$.
- In consequence, the quotient module is isomorphic to the weighted Bergman module $\mathscr{H}^{(\nu)}(\mathbb{B}_n)$ and these are essentially reductive.

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Arveson-Douglas Conjecture

Lemma (Arveson). Let $\mathscr{M}$ be an essentially reductive Hilbert module over $\mathbb{C}[z]$, $\mathscr{N}$ be a submodule of $\mathscr{M}$ and $\mathscr{Q} = \mathscr{M} \ominus \mathscr{N}$, be the corresponding quotient module. The submodule $\mathscr{N}$ is essentially reductive if and only if so is $\mathscr{Q}$.

Arveson-Douglas Conjecture. Let $\mathscr{M}$ be a essentially reductive Hilbert module over the algebra $\mathbb{C}[z]$ consisting of holomorphic functions defined on $\Omega \subseteq \mathbb{C}^m$. If the ideal I is homogeneous, that is, generated by homogeneous polynomials, then the submodule $[I]$ is r -essentially reductive for every $r > \dim V(I)$.

A positive solution to the Arveson-Douglas conjecture implies that

$$0 \to \mathscr{C}([I]) \to \mathscr{T}([I]) \to C(V(I) \cap \partial\mathbb{B}_m) \to 0$$

is a short exact sequence. One of the main questions is to identify the K - homology class $[\mathscr{T}([I])]$ defined by this extension. In the case that $I = \{0\}$ and $m = 1$, the Toeplitz index theorem for $\mathbb{T}$ gives the answer to this question.

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tensor product

Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be Hilbert spaces of holomorphic functions on Ω so that they possess reproducing kernels K_1 and $K_2,$ respectively. Assume that the natural action of $\mathbb{C}[\underline{z}]$ on the Hilbert space $\mathscr{M}_1$ is continuous, that is, the map $(p,h) \to ph$ defines a bounded operator on M_p for $p \in \mathbb{C}[\underline{z}].$ (We make no such assumption about the Hilbert space $\mathscr{M}_2.$) Now, $\mathbb{C}[\underline{z}]$ acts naturally on the Hilbert space tensor product $\mathscr{M}_1 \otimes \mathscr{M}_2$ via the map

$$(p,(h \otimes k)) \to ph \otimes k, p \in \mathbb{C}[\underline{z}],\ h \in \mathscr{M}_1,\ k \in \mathscr{M}_2.$$

The map $h \otimes k \to hk$ identifies the Hilbert space $\mathscr{M}_1 \otimes \mathscr{M}_2$ as a reproducing kernel Hilbert space of holomorphic functions on $\Omega \times \Omega.$ The module action is then the point-wise multiplication $(p,hk) \to (ph)k,$ where $((ph)k)(z_1,z_2) = p(z_1)h(z_1)k(z_2),\ z_1,z_2 \in \Omega.$

a new kernel

Let $\mathscr{H}$ be the Hilbert module $\mathscr{M}_1 \otimes \mathscr{M}_2$ over $\mathbb{C}[\underline{z}]$. Let $\triangle \subseteq \Omega \times \Omega$ be the diagonal subset $\{(z,z) : z \in \Omega\}$ of $\Omega \times \Omega$. Let $\mathscr{S}$ be the maximal submodule of functions in $\mathscr{M}_1 \otimes \mathscr{M}_2$ which vanish on $\triangle$. Thus

$$0 \to \mathscr{S} \xrightarrow{X} \mathscr{M}_1 \otimes \mathscr{M}_2 \xrightarrow{Y} \mathscr{Q} \to 0$$

is a short exact sequence, where $\mathscr{Q} = (\mathscr{M}_1 \otimes \mathscr{M}_2)/\mathscr{S}$, X is the inclusion map and Y is the natural quotient map. One can appeal to an extension of an earlier result of Aronszajn to analyze the quotient module $\mathscr{Q}$ when the given modules are reproducing kernel Hilbert spaces. The reproducing kernel of $\mathscr{H}$ is then the pointwise product $K_1(z,w)K_2(u,v)$ for $z,w;u,v$ in Ω. Set $\mathscr{H}_{\text{res}} = \{f_{|\triangle} : f \in \mathscr{H}\}$ and $\|f\|_{|\triangle} = \inf\{\|g\| : g \in \mathscr{H}, g_{|\triangle} \equiv f_{|\triangle}\}$.

- The quotient module is isomorphic to the module $\mathscr{H}_{\text{res}}$ whose reproducing kernel is the pointwise product $K_1(z,w)K_2(z,w)$, $z,w \in \Omega$.

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another kernel!

Suppose $\Omega \subseteq \mathbb{C}^d$ is open connected and bounded. Let $K : \Omega \times \Omega$ be a non-negative definite kernel. Then $\widetilde{K}$ defined by

$$\widetilde{K}(z,w) = \left(\!\left(K^2 \partial_i \bar{\partial}_j \log K(z,w)\right)\!\right)_{1 \leq i,j \leq d}$$

is a $\mathbb{C}^{d \times d}$ valued non-negative definite kernel.

- We point out that $\sum_{i,j} \partial_i \bar{\partial}_j \log K(w,w) dw_i \wedge d\bar{w}_j$ is the curvature of the metric $K(w,w)$.

  To see that $\widetilde{K}$ defines a positive definite kernel on Ω, set

$$\phi_i(w) := K_w \otimes \bar{\partial}_i K_w - \bar{\partial}_i K_w \otimes K_w, 1 \leq i \leq m$$

and note that each $\phi : \Omega \to \mathscr{H}$ is holomorphic. A simple calculation then shows that

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what is the Hilbert module?

How to describe the Hilbert space, or more importantly, the Hilbert module $\mathscr{H}(\widetilde{K})$? May be, it is a quotient of the Hilbert module $\mathscr{H} \otimes \mathscr{H}$? If so, How do we identify the corresponding submodule?

Let $\mathscr{H}_0$ be the subspace of $\mathscr{H}(K) \otimes \mathscr{H}(K)$ given by $\overline{\bigvee}\{\phi_i(w) : w \in \Omega, 1 \leq i \leq m\}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the submodules defined by

$$\mathscr{H}_1 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = 0\}$$

and

$$\mathscr{H}_2 = \{f \in \mathscr{H}(K) \otimes \mathscr{H}(K) : f|_\Delta = \partial_1 f|_\Delta = \partial_2 f|_\Delta = \ldots = \partial_m f|_\Delta = 0\}.$$

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How to describe the Hilbert space, or more importantly, the Hilbert module $\mathscr{H}(\widetilde{K})$? May be, it is a quotient of the Hilbert module $\mathscr{H}\otimes\mathscr{H}$? If so, How do we identify the corresponding submodule?

Let $\mathscr{H}_0$ be the subspace of $\mathscr{H}(K)\otimes\mathscr{H}(K)$ given by $\overline{\bigvee}\{\phi_i(w) : w\in\Omega, 1\leq i\leq m\}$.

From this definition, it is not clear which functions belong to the subspace. We give an explicit description.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the submodules defined by

$$\mathscr{H}_1 = \{f\in\mathscr{H}(K)\otimes\mathscr{H}(K) : f|_\Delta = 0\}$$

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example

Let $\mathscr{H}^{(\lambda)}$ be the Hilbert module over the disc algebra determined by the kernel $K^{(\lambda)}(z,w) := (1 - z\bar{w})^{-2\lambda}$. We identify the tensor product $\mathscr{H}^{(\lambda,\mu)}(\mathbb{D}^2) := \mathscr{H}^{(\lambda)} \otimes \mathscr{H}^{(\mu)}$ as a vector subspace of $Hol(\mathbb{D} \times \mathbb{D})$. Set

$$\mathscr{H}_k^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in \mathscr{H}^{(\lambda,\mu)}(\mathbb{D}^2) : f_{|\triangle} = (\partial_1 f)_{|\triangle} = \cdots (\partial_1^{k-1} f)_{|\triangle} = 0\},$$

where $\triangle := \{(z,z) : z \in \mathbb{D}\}$.

Set

$$\mathscr{Q}_k := \mathscr{H}_k^{(\lambda,\mu)}(\mathbb{D}^2) \ominus \mathscr{H}_{k+1}^{(\lambda,\mu)}(\mathbb{D}^2).$$

The reproducing kernel for $\mathscr{Q}_1$ is $K^{(\lambda+\mu)}$, while that of $\mathscr{Q}_2$ is $K^{(\lambda+\mu+2)}$.

- This is the Clebsch-Gordan formula!

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Thank You!