

# Adventures in the Bergman Land

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Workshop on Non-commutative Analysis and Probability

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based on conversations with

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# The Cowen-Douglas class

Let  $T = (T_1, \dots, T_m)$  acting on a complex separable Hilbert space  $\mathcal{H}$  be a commuting tuple of bounded linear operators possessing an open set  $\Omega$  of (joint) eigenvalues of constant multiplicity.

Assuming that the linear span of the eigenvectors  $\gamma(w)$ ,  $w \in \Omega$ , is dense in  $\mathcal{H}$  provides a model for such operators, namely, they can be realized as the adjoint of the  $m$ -tuple of multiplication operators  $M = (M_{z_1}, \dots, M_{z_m})$  on some reproducing kernel Hilbert space  $\mathcal{H}_K$  of holomorphic functions defined on  $\Omega$  taking values in  $\mathbb{C}^n$ , where  $n$  is the constant multiplicity of the eigenvalues.



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The commuting tuple  $M$  defines a module multiplication on  $\mathcal{H}$  over the polynomial ring:

$$(p, h) \mapsto M_p h, \quad p \in \mathbb{C}[z], \quad h \in \mathcal{H},$$

where  $(M_p h)(z) = p(z)h(z)$  and conversely.

The commuting tuple defines a holomorphic Hermitian vector bundle  $E_T$  on  $\Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$  by choosing the fiber of  $E_M$  at  $w$  to be the eigenspace of  $M^*$  at  $w$ . Note all these fibers are a subspace of a fixed Hilbert space  $\mathcal{H}$ .

Theorem (Cowen-Douglas) Two commuting tuples  $T$  and  $\hat{T}$  are unitarily equivalent if and only if the vector bundles  $E_T$  and  $E_{\hat{T}}$  are locally equivalent via a holomorphic bundle map.



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**Theorem (Cowen-Douglas)** Two commuting tuples  $T$  and  $\tilde{T}$  are unitarily equivalent if and only if the vector bundles  $E_T$  and  $E_{\tilde{T}}$  are locally equivalent via a holomorphic bundle map.



# homogeneity

Let  $G$  be a locally compact second countable group acting on  $\Omega$  transitively. Thus,  $\Omega$  is a  $G$ -space and assuming that the action is continuous, it is of the form  $G/H$ , for some closed subgroup  $H$  of  $G$ .

The examples will be explicit consisting of the Euclidean ball  $\mathbb{B}_m$  and the bi-holomorphic automorphism group of  $\mathbb{B}_m$ .

If this action lifts to the vector bundle  $E_T$ , that is,  $g \in G$  also defines an automorphism  $\hat{g}$  of  $E_T \xrightarrow{\pi} \Omega$  such that  $\pi \hat{g} = g \pi$ , then the vector bundle  $E_T$  is equivariant, or equivalently, via the Cowen-Douglas theorem the operator  $T$  is homogeneous, namely,  $g(T)$  is unitarily equivalent to  $T$ .

It is a special feature of this class of operators that a choice of the intertwining ( $T$  and  $g(T)$ ) unitary  $U_g$  exists so that the map  $g \mapsto U_g$  is strongly continuous.



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# refinement of the Cowen-Douglas theorem

For simplicity, we only consider the case  $n=2$ . Thus, the dimension of the eigenspace at  $w \in \Omega$  is 2.

Even for a single operators  $T \in B_2(\Omega)$  extracting explicit invariants from the Cowen-Douglas theorem is not easy. Therefore, looking for a smaller class of operators is worthwhile.

If  $m=1$ , then for any  $T \in B_2(\Omega)$  acting on a Hilbert space  $\mathcal{H}$ , there exists a decomposition  $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_C$ , where the symbol  $\oplus$  stands for the orthogonal direct sum of the two Hilbert space, such that

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, A, B \in B_1(\Omega), S: \mathcal{H}_1 \rightarrow \mathcal{H}_2.$$

Now, impose the condition  $AB = BA$ .

This ensures the existence of holomorphic flag  $F \in E$  compatible with the module action.



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the meaning of all this

Roughly speaking, the second fundamental form of this inclusion along with the curvature of  $E_T$  forms a complete set of unitary invariant for  $T$ .

The intertwining condition we impose also defines a cocycle via  $B$ . The module map is

$$(p, h) \mapsto m_p h := \begin{pmatrix} p(A) & Bp'(C) \\ 0 & p(C) \end{pmatrix} h, \quad p \in \mathbb{C}[z], h \in \mathcal{H}.$$

Here, the cocycle  $\sigma: \mathbb{C}[z] \times \mathcal{H}_C \rightarrow \mathcal{H}_A$ , is defined as:

$(p, \cdot) \mapsto Bp'(C)$ . Without the intertwining condition, the functional calculus is much more complicated.

Equivalently, the short sequence of Hilbert modules over the polynomial ring

$$0 \rightarrow \mathcal{H}_A \xrightarrow{L} \mathcal{H} \xrightarrow{\pi} \mathcal{H}_C \rightarrow 0$$

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## commuting tuples in the CD class

Let  $\mathbf{T} = (T_1, \dots, T_m) \in B_2(\Omega)$ . We say that  $T$  admits a flag if there exist:

(i) two commuting tuples  $\mathbf{A} = (A_1, \dots, A_m) \in B_1(\Omega)$ , and  $\mathbf{C} = (C_1, \dots, C_m) \in B_1(\Omega)$ ;

(ii) bounded operators  $S_1, \dots, S_m : \mathcal{H}_C \rightarrow \mathcal{H}_A$ , satisfying the intertwining condition

$$A_i S_j = S_j C_i, \quad 1 \leq i, j \leq m,$$

with respect to some decomposition  $\mathcal{H} = \mathcal{H}_A \tilde{\oplus} \mathcal{H}_C$ ,

$$T_k = \begin{pmatrix} A_k & S_k \\ 0 & C_k \end{pmatrix}, \quad k = 1, \dots, m.$$

Define  $\sigma_{z_i} : \mathcal{H}_C \rightarrow \mathcal{H}_A$  by  $\sigma_{z_i} = S_i$ . Then the cocycle condition is

$$\sigma_{z_i} C_j - A_i \sigma_{z_j} = \sigma_{z_j} C_i - A_j \sigma_{z_i}, \quad 1 \leq i, j \leq m.$$

This is the first-order cocycle condition for extending the module structures from  $\mathcal{H}_A$  and  $\mathcal{H}_C$  to the direct sum  $\mathcal{H} = \mathcal{H}_A \tilde{\oplus} \mathcal{H}_C$ .



## the module compatibility

What about compatibility with the module action? If we pick a polynomial  $p$ , it is clear that

$$p(T) = \begin{pmatrix} p(A) & \sigma(p) \\ 0 & p(C) \end{pmatrix}$$

where  $\sigma$  must be a cocycle. So, setting,  $\sigma(z_i) = S_i$ , and defining  $\sigma_p$  by the requirement

$$\sigma(pq) = p(A)\sigma(q) + \sigma(p)p(C),$$

it is easy to verify that  $\sigma_p$  defined recursively is well defined by the commutativity of the tuple  $T$ .

The commutativity of the tuple  $T$  actually follows from the intertwining condition imposed by the requirement of a flag.

A simple computation actually gives:  $p(T)_{12} = \sum_{k=1}^m S_k \frac{\partial p}{\partial z_k}(C_k)$   
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The set of all extensions is parametrized (this is just the commutativity of  $T$ ) by:

$$\{(S_1, \dots, S_m) : A_i S_j - S_j C_i = A_j S_i - S_i C_j, 1 \leq i, j \leq m\}.$$

This is an affine space (not a vector space in general). The intertwining condition picks out a subspace, namely,  $A_i S_j = S_j C_i$ ,  $1 \leq i, j \leq m$ . This defines a linear subspace of all possible extensions.

The general solutions  $s$  that form a cocycle is an affine space. But the linear subspace satisfying the intertwining conditions, we expect ensures the extension is nonsplit (or equivalently, strongly irreducible) if  $s$  is not a coboundary. A coboundary  $s$  is of the form  $s_i = A_i X - X C_i$ ,  $1 \leq i \leq m$ , for some  $X : \mathcal{H}_C \rightarrow \mathcal{H}_A$ . This was proved for  $m=1$  earlier.

An extension can fail to split without satisfying the strong intertwining condition!



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An extension can fail to split without satisfying the strong intertwining condition!



# Cowen-Douglas bundle and the flag

The Cowen-Douglas bundle  $E_T$  fits into a short exact sequence of holomorphic Hermitian vector bundles:

$$0 \rightarrow E_A \rightarrow E_T \rightarrow E_C \rightarrow 0.$$

Let  $\beta \in \Omega^{1,0}(\Omega, \text{Hom}(Q, F))$ . Write  $\beta = \sum_{k=1}^m \beta_k dz_k$ . We require  $\beta_k \in \text{Hom}_\theta(Q, F)$  to be holomorphic bundle maps for each  $k$ .

The intertwining relations  $A_i S_j = S_j C_i$  say that each  $\beta_j$  is a module map between the Cowen-Douglas line bundles  $E_C \rightarrow E_A$ .

For each  $w \in \Omega$  and  $h_C \in E_C(w)$ , we have

$$E_T(w) = \left\{ \begin{pmatrix} h_A + \beta_w(h_C) \\ h_C \end{pmatrix} : h_A \in E_A(w), h_C \in E_C(w) \right\}.$$



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## the definition of a flag is intrinsic

In Carlson-Clark language, an extension is given by a cocycle  $\sigma: \mathbb{A} \times \mathcal{H}_C \rightarrow \mathcal{H}_A$  with values  $\sigma(z_k, \cdot) = S_k$  on generators.

Now, changing the splitting (i.e. choosing a different bounded right inverse of the quotient map) changes  $\sigma$  by a coboundary:  $\sigma \mapsto \sigma + \delta X$ , where

$$(\delta X)(a, k) = a \cdot Xk - X(a \cdot k).$$

So the extension class is the equivalence class  $[\sigma]$ , not a specific choice of  $\sigma$ .

Thus the intertwining condition is equivalent to the statement: the extension class  $[\sigma]$  admits a representative with  $\sigma(z_k, \cdot) = S_k$  and  $A_i S_k = S_k C_i$ ,  $1 \leq i, k \leq m$ . This depends only on the class  $[\sigma]$ , not on the splitting. The representative is not unique, but the existence of such a representative is invariant.

In other words, the extension class is represented by a holomorphic 1-cocycle with values in  $\text{Hom}(Q, F)$ .



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Now, changing the splitting (i.e. choosing a different bounded right inverse of the quotient map) changes  $\sigma$  by a coboundary:  $\sigma \mapsto \sigma + \delta X$ , where

$$(\delta X)(a, k) = a \cdot Xk - X(a \cdot k).$$

So the extension class is the equivalence class  $[\sigma]$ , not a specific choice of  $\sigma$ .

Thus the intertwining condition is equivalent to the statement: the extension class  $[\sigma]$  admits a representative with  $\sigma(z_k, \cdot) = S_k$  and  $A_i S_k = S_k C_i$ ,  $1 \leq i, k \leq m$ . This depends only on the class  $[\sigma]$ , not on the splitting. The representative is not unique, but the existence of such a representative is invariant.

In other words, the extension class is represented by a holomorphic 1-cocycle with values in  $\text{Hom}(Q, F)$ .



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## examples, the Bergman metric

For  $s > 0$ , the kernel  $B^{(s)}(z, w) = (1 - \langle z, w \rangle)^{-s}$ ,  $z, w \in \mathbb{B}_n$  is positive definite. Let

$$\mathcal{K}(w) := \left( \left( \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log B^{(s)}(w, w) \right) \right)_{i,j=1}^n.$$

In this case, that is, where the ordinary Wallach set has no discrete points, it can be shown that  $\mathcal{K}^{(s)}(w)$  also defines a positive definite kernel on  $\mathbb{B}_n$ .

We have the following transformation rule for  $\mathcal{K}^{(s)}(w)$  matching that of the kernel  $B^{(s)}$ , namely,

$$\mathcal{K}^{(s)}(w) = D\varphi(w) \mathcal{K}^{(s)}(\varphi(w))' \overline{D\varphi(w)},$$

where  $\varphi$  is any bi-holomorphic automorphism of the ball  $\mathbb{B}_n$  and  $D\varphi(w)$  is the complex derivative of  $\varphi$ . Thus,  $\mathcal{K}$  according to a natural cocycle, namely,  $(\varphi, w) = D\varphi(w)$ .



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# the explicit computation

Evidently, we have

$$\mathcal{K}(w) = \frac{s}{(1 - \|w\|^2)^2} \begin{cases} w_i \bar{w}_j & \text{if } i \neq j \\ 1 - \sum_{\substack{k=1 \\ k \neq i}}^n |w_k|^2 & \text{if } i = j. \end{cases}$$

Moreover,  $(1 - \|w\|^2)^{-s} \mathcal{K}(w)$  will define a positive definite kernel on  $\mathbb{B}_n$  for any  $s > 0$  independent of  $n$ . Let  $\mathcal{H}^{(s)}$  be the Hilbert space of holomorphic functions defined on  $\mathbb{B}_n$  uniquely defined by the reproducing kernel  $(1 - \|w\|^2)^{-s} \mathcal{K}^{(s)}(w)$ . Clearly,  $\det(D\varphi(w))^s D\varphi(w)$  is also a cocycle, therefore, the map

$$(U_{\varphi^{-1}} f)(z) = (\det D\varphi(w))^s D\varphi(w) (f \circ \varphi)(w), \quad f \in \mathcal{H},$$

defines a unitary homomorphism from the biholomorphic automorphism group of  $\mathbb{B}_n$  to the unitary group on the Hilbert space  $\mathcal{H}_{\mathbb{K}^{(s)}}$ , where

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# action of the unitary group

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- If  $g$  is unitary, then it acts irreducibly on the fiber at 0, which is isomorphic to  $\mathbb{C}^n$ .
- This means that the algebra generated by the co-efficients of the curvature is the full matrix algebra and consequently, the homogeneous bundle  $E$  is irreducible with respect to the Hermitian structure induced by the Bergman metric.

For simplicity, taking  $n=2$ , and note that the curvature co-efficients of the metric at  $(0,0)$  consists of

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## how do we get non-trivial flags?

This does not disprove the existence of a holomorphic flag. However, the off-diagonal operators can be computed relative to any choice of a holomorphic subbundle. They are multiplication operators and cannot possess the intertwining property since the diagonal elements are shifts.

The answer is to put together two Hilbert modules and find the cocycle to ensure compatibility. This is the jet construction that appears in several different situations. To describe it briefly, let us assume  $n = 2$ . Then the Bergman metric  $\mathbb{K}^{(s)}$  defines a Hermitian structure for a holomorphic vector bundle of rank 2. The Bergman kernel, or a power of the Bergman kernel  $B^{(s)}(z, w) = (1 - \langle z, w \rangle)^{-s}$  defines a line bundle  $\mathcal{L}^{(s)}$ . For any  $f \in \mathcal{H}_{B^{(s)}}$ , consider the map

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## putting it all together

Now, transform the orthonormal basis of  $\mathcal{H}_{B^{(s)}}$  via  $\Gamma$  to an orthonormal basis in the space of holomorphic functions defined on  $\mathbb{B}_2$  taking values in  $\mathbb{C}^3$ .

Let  $\mathbb{H}^{(s)}$  denote the Hilbert space obtained by taking the linear span of these basis vectors and then completing with respect to the inner product  $\langle \Gamma f, \Gamma g \rangle := \langle f, g \rangle$ .

so, we have all the ingredients for the construction of a non-trivial flag.

Consider the Hilbert space  $H^{(s)}$  uniquely determined by the sum  $K_{\Gamma} + B^{(s)}$ , or equivalently, the rank 3 holomorphic Hermitian vector bundle defined by the Hermitian structure:  $K_{\Gamma} + B^{(s)}$ . The rank 2 bundle sits inside this rank 3 holomorphic bundle as a holomorphic sub-bundle



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This gives a simultaneous triangular form for the pair of multiplication operators on the Hilbert space

$$H^{(s)} = \mathcal{H}_T^{(s)} \oplus \mathbb{H}^{(s)},$$

and consequently, we get a cocycle, and a holomorphically induced representation that goes with it!



Thank You!

