Flag structure for operators in the Cowen-Douglas class

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The Cowen-Douglas class $B_n(\Omega)$ consists of those bounded linear operators T on a complex separable Hilbert space $\mathcal{H}$ which possess an open set $\Omega \subset \mathbb{C}$ of eigenvalues of constant multiplicity n and admit a holomorphic choice of eigenvectors: $s_1(w), \ldots, s_n(w)$, $w \in \Omega$, in other words, there exists holomorphic functions $s_1, \ldots, s_n : \Omega \to \mathcal{H}$ which span the eigenspace of T at $w \in \Omega$.

The holomorphic choice of eigenvectors $s_1, \ldots, s_n$ defines a holomorphic Hermitian vector bundle E_T via the map

$$s : \Omega \to \mathrm{Gr}(n, \mathcal{H}),\ s(w) = \ker(T - w) \subseteq \mathcal{H}.$$

In the paper [3], Cowen and Douglas show that there is a one to one correspondence between the unitary equivalence class of the operators T in $B_n(\Omega)$ and the equivelence classes of the holomorphic Hermitian vector bundles E_T determined by them. They also find a set of complete invariants for this equivalence consisting of the curvature $\mathcal{K}$ of E_T and a certain number of its covariant derivatives. Unfortunately, these invariants are not easy to compute unless n is 1.

Finding similarity invariants for operators in the class $B_n(\Omega)$ has been somewhat difficult from the beginning. The conjecture made by Cowen and Douglas in [3] was shown to be false [1, 2]. However, significant progress on the question of similarity has been made recently (cf. [6, 9]).

We isolate a subset of irreducible operators in the Cowen-Douglas class $B_n(\Omega)$ for which a complete set of tractable unitary invariants is relatively easy to identify. We also determine when two operators in this class are similar.

We introduce below this smaller class $\mathcal{F}B_2(\Omega)$ of operators in $B_2(\Omega)$ leaving out the more general definition for now.

Definition 1. *We let $\mathcal{F}B_2(\Omega)$ denote the set of bounded linear operators T for which we can find operators T_0, T_1 in $\mathcal{B}_1(\Omega)$ and an intertwiner S between T_0 and T_1, that is, $T_0 S = S T_1$ so that*

$$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}.$$

An operator T in $B_2(\Omega)$ admits a decomposition of the form (cf. [9, Theorem 1.49, pp. 48]) $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some pair of operators T_0 and T_1 in $B_1(\Omega)$. Conversely, an operator T, which admits a decomposition of this form for some choice of T_0, T_1 in $B_1(\Omega)$ is seen to be in $B_2(\Omega)$. In defining the new class $\mathcal{F}B_2(\Omega)$, we are merely imposing one additional condition, namely that $T_0 S = S T_1$.

We show that T is in the class $\mathcal{F}B_2(\Omega)$ if and only if there exist a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\gamma_0(w)$ and $t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω. This is also equivalent to the existence of a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle$, $w \in \Omega$. Our first main theorem on unitary classification is given below.

THEOREM 1. *Let* $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ *and* $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ *be two operators in* $\mathcal{F}B_2(\Omega)$. *Also let* t_1 *and* $\tilde{t}_1$ *be non-zero sections of the holomorphic Hermitian vector bundles* E_{T_1} *and* $E_{\tilde{T}_1}$ *respectively. The operators* T *and* $\tilde{T}$ *are equivalent if and only if* $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$ *(or* $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$*) and* $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$.

Cowen and Douglas point out in [3] that an operator in $B_1(\Omega)$ must be irreducible. However, determining which operators in $B_n(\Omega)$ are irreducible is a formidable task. It turns out that the operators in $\mathcal{F}B_2(\Omega)$ are always irreducible. Indeed, if we assume S is invertible, then T is strongly irreducible.

Recall that an operator T in the Cowen-Douglas class $B_n(\Omega)$, up to unitary equivalence, is the adjoint of the multiplication operator M on a Hilbert space $\mathcal{H}$ consisting of holomorphic functions on $\Omega^* := \{\bar{w} : w \in \Omega\}$ possessing a reproducing kernel K. What about operators in $\mathcal{F}B_n(\Omega)$? For $n = 2$, a model for these operators is described below.

Let $\gamma = (\gamma_0, \gamma_1)$ be a holomorphic frame for the vector bundle E_T, $T \in \mathcal{F}B_2(\Omega)$. Then the operator T is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_\Gamma \subseteq \mathrm{Hol}(\Omega^*, \mathbb{C}^2)$ possessing a reproducing kernel $K_\Gamma : \Omega^* \times \Omega^* \to \mathbb{C}^{2\times 2}$. It is easy to write down the kernel K_Γ explictly: For $z, w \in \Omega^*$, we have

$$\begin{aligned} K_\Gamma(z,w) &= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_0(\bar{z})\rangle \\ \langle \gamma_0(\bar{w}), \gamma_1(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_1(\bar{z})\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial}{\partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle \\ \frac{\partial}{\partial z}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial^2}{\partial z \partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle + \langle t_1(\bar{w}), t_1(\bar{z})\rangle \end{pmatrix}, \end{aligned}$$

where t_1 and $\gamma_0 := S(t_1)$ are frames of the line bundles E_{T_1} and E_{T_0} respectively. It follows that $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and that $t_1(w)$ is orthogonal to $\gamma_0(w)$, $w \in \Omega$.

Setting $K_0(z,w) = \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle$ and $K_1(z,w) = \langle t_1(\bar{w}), t_1(\bar{z})\rangle$, we see that the reproducing kernel K_Γ has the form:

$$K_\Gamma(z,w) = \begin{pmatrix} K_0(z,w) & \frac{\partial}{\partial \bar{w}} K_0(z,w) \\ \frac{\partial}{\partial z} K_0(z,w) & \frac{\partial^2}{\partial z \partial \bar{w}} K_0(z,w) + K_1(z,w) \end{pmatrix}. \tag{1}$$

We now give examples of natural classes of operators that belong to $\mathcal{F}B_2(\Omega)$. Indeed, we were led to the definition of this new class $\mathcal{F}B_2(\Omega)$ of operators by trying to understand these examples better.

An operator T is called *homogeneous* if $\phi(T)$ is unitarily equivalent to T for all ϕ in Möb which are analytic on the spectrum of T.

If an operator T is in $\mathcal{B}_1(\mathbb{D})$, then T is homogeneous if and only if $\mathcal{K}_T(w) = -\lambda(1-|w|^2)^{-2}$, for some $\lambda > 0$. The paper [10] provides a model for all homogeneous operators in $B_n(\mathbb{D})$. We describe them for $n = 2$. For $\lambda > 1$ and $\mu > 0$, set $K_0(z,w) = (1 - z\bar{w})^{-\lambda}$ and $K_1(z,w) = \mu(1 - z\bar{w})^{-\lambda-2}$. An irreducible operator T in $B_2(\mathbb{D})$ is homogeneous if and only if it is unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D}, \mathbb{C}^2)$ determined by the positive definite kernel given in equation (1). The similarity as well as a unitary classification of homogeneous operators in $B_n(\mathbb{D})$ were obtained in [10] using non-trivial results from representation theory of semi-simple Lie group. For $n = 2$, this classification is a consequence of Theorem 1.

An operator T in $B_1(\Omega)$ acting on a Hilbert space $\mathcal{H}$ makes it a module over the polynomial ring via the usual point-wise multiplication. An important tool in the study of these modules is the localization. This is the Hilbert module $J\mathcal{H}^{(k)}_{\rm loc}$ corresponding to the spectral sheaf $J\mathcal{H} \otimes_{\mathcal{P}} \mathbb{C}^k_w$, where

(1) $\mathcal{P}$ is the polynomial ring,
(2) $\mathbb{C}^k_w$ is a k - dimensional module over the polynomial ring,
(3) the module action on $\mathbb{C}^k_w$ is via the map $\mathcal{J}(w)$, see [7, (2.8) pp. 376];
(4) $J : \mathcal{H} \to \mathrm{Hol}(\Omega, \mathbb{C}^k)$ is the jet map, namely, $Jf = \sum_{\ell=0}^{k-1} \partial^\ell f \otimes \varepsilon_{\ell+1}$, $\varepsilon_1, \ldots, \varepsilon_k$ are the standard unit vectors in $\mathbb{C}^k$.

We now consider the localization with $k = 2$. If we assume that the operator T has been realized as the adjoint of the multiplication operator on a Hilbert space of holomorphc function possessing a kernel function, say K, then the kernel $JK^{(2)}_{\rm loc}$ for the localization (of rank 2) given in [7, (4.2) pp. 393] coincides with K_Γ of equation (1). In this case, we have $K_1 = K = K_0$.

As is to be expected, using the complete set of unitary invariants given in Theorem 1, we see that the unitary equivalence class of the Hilbert module $\mathcal{H}$ is in one to one correspondence with that of $J\mathcal{H}^{(2)}_{\rm loc}$.

Thus the class $\mathcal{F}B_2(\Omega)$ contains two very interesting classes of operators. For $n > 2$, we find that there are competing definitions. One of these contains the homogeneous operators and the other contains the Hilbert modules obtained from the localization.

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