Basics Of Graph Morphology

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Granulometries

Why Discrete Mathematical Morphology?

- Superior Analysis
  - Finer Granulometries
  - Contrast Preserving Watershed Algorithms
- Fast Graph based computations

Most of the material is available in [1],[3],[2]

Basics of Graphs

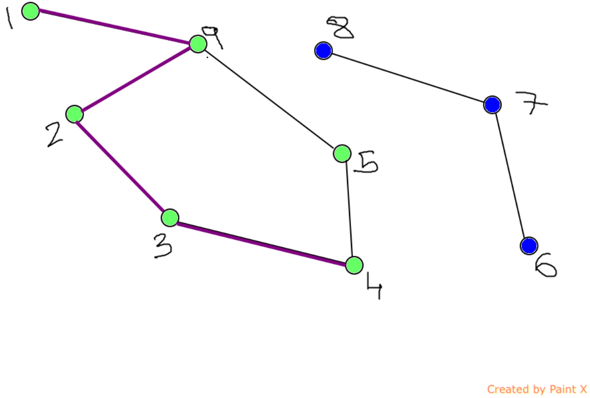

Figure : General Graph

- Definition of a Graph
- Adjacency

Basics of Graphs

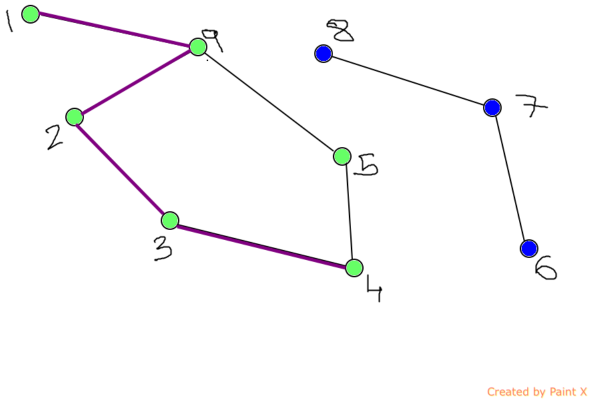

Figure : General Graph

- Paths in a Graph
- Connected Components
- Vertex and Edge Weighted Graphs

Graphs in Graph Morphology

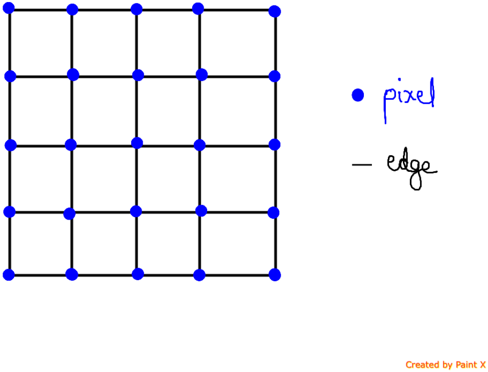

Figure : 4 - Adjacency Graph

Graphs in Graph Morphology

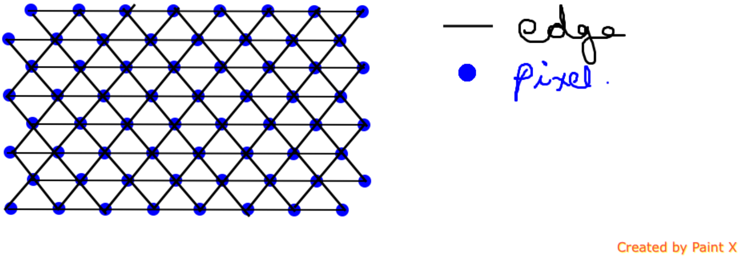

Figure : 6 - Adjacency Graph

Basic Operators on Graphs

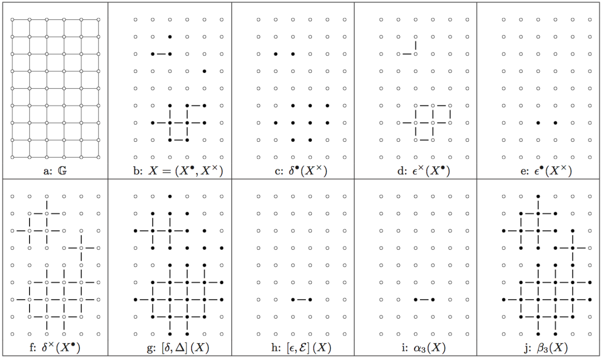

Figure : Graph Dilation And Erosion

$$\delta^\bullet(X^\times) = \left\{x \in \mathbb{G}^\bullet \mid \exists e_{x,y} \in X^\times\right\}$$

Basic Operators on Graphs

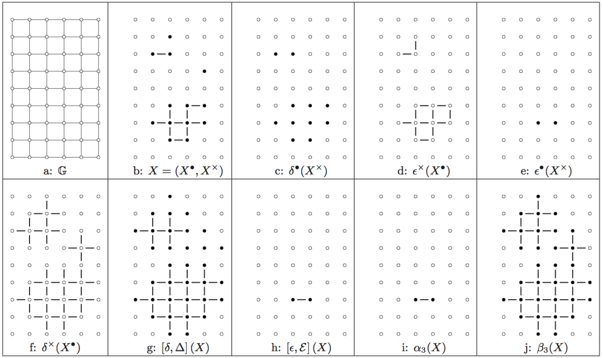

Figure : Graph Dilation And Erosion

$$\epsilon^\times(X^\bullet) = \left\{ e_{x,y} \in \mathbb{G}^\times \mid x \in X^\bullet \textit{ and } y \in X^\bullet \right\}$$

Basic Operators on Graphs

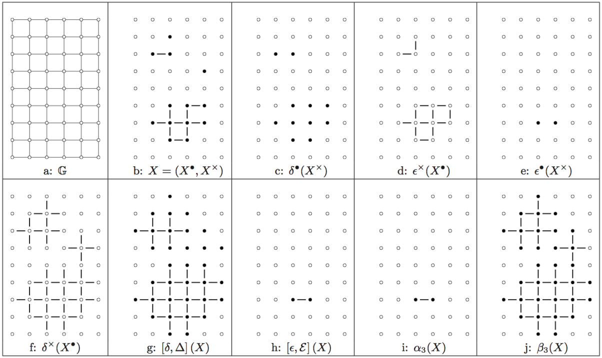

Figure : Graph Dilation And Erosion

$$\epsilon^\bullet(X^\times) = \left\{x \in \mathbb{G}^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times, e_{x,y} \in X^\times\right\}$$

Basic Operators on Graphs

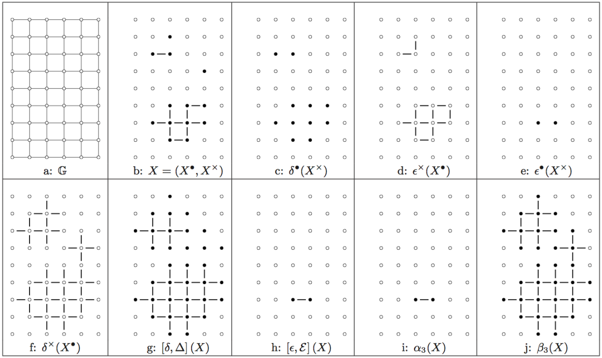

Figure : Graph Dilation And Erosion

$$\delta^\times(X^\bullet) = \left\{ e_{x,y} \in \mathbb{G}^\times \mid x \in X^\bullet \ or \ y \in X^\bullet \right\}$$

Vertex Dilation and Erosion

Definition

We define the notion of **vertex dilation**, δ and **vertex erosion**, ϵ as, $\delta = \delta^\bullet \circ \delta^\times$ and $\epsilon = \epsilon^\bullet \circ \epsilon^\times$. These are equivalent to, for any $X^\bullet \in \mathbb{G}^\bullet$

$$\begin{aligned}
\delta(X^\bullet) &= \left\{x \in \mathbb{G}^\bullet \mid \exists e_{x,y} \in X^\times,\ e_{x,y} \cap X^\bullet \neq \phi\right\} \\
\epsilon(X^\bullet) &= \left\{x \in \mathbb{G}^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times,\ x, y \in X^\bullet\right\}
\end{aligned}$$

Edge Dilation and Erosion

Definition

We define the notion of **edge dilation**, Δ and **edge erosion**, $\mathcal{E}$ as, $\Delta = \delta^{\times} \circ \delta^{\bullet}$ and $\mathcal{E} = \epsilon^{\times} \circ \epsilon^{\bullet}$. These are equivalent to, for any $X^{\times} \in \mathbb{G}^{\bullet}$

$$
\begin{aligned}
\Delta(X^{\times}) &= \left\{ e_{x,y} \in \mathbb{G}^{\times} \mid \ \textit{either } \exists e_{x,z} \in X^{\times} \ \textit{or} \ e_{y,w} \in X^{\times} \right\} \\
\mathcal{E}(X^{\times}) &= \left\{ e_{x,y} \in \mathbb{G}^{\times} \mid \forall e_{x,z} \ e_{y,w} \in \mathbb{G}^{\times}, \ e_{x,z} \in X^{\times}, \ e_{y,w} \in X^{\times} \right\}
\end{aligned}
$$

Vertex Dilation

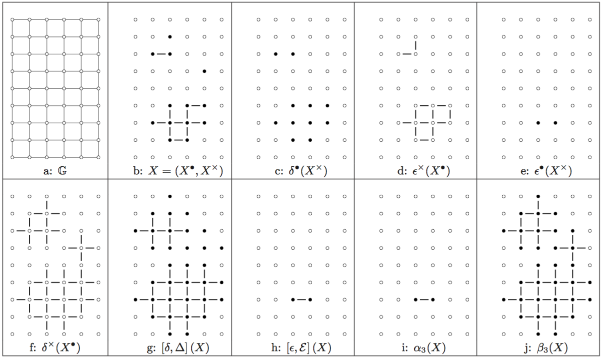

Figure : Graph Dilation And Erosion

$$\delta(X^{\bullet}) = \{x \in \mathbb{G}^{\bullet} \mid \exists e_{x,y} \in \mathbb{G}^{\times},\ e_{x,y} \cap X^{\bullet} \neq \phi\}$$

Edge Dilation

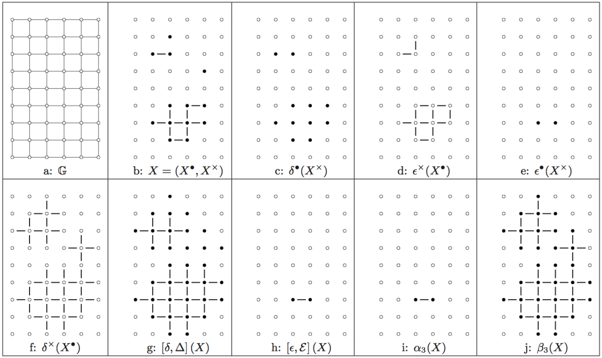

Figure : Graph Dilation And Erosion

$$\Delta(X^{\times}) = \left\{ e_{x,y} \in \mathbb{G}^{\times} \mid \textit{ either } \exists e_{x,z} \in X^{\times} \textit{ or } e_{y,w} \in X^{\times} \right\}$$

Vertex Erosion

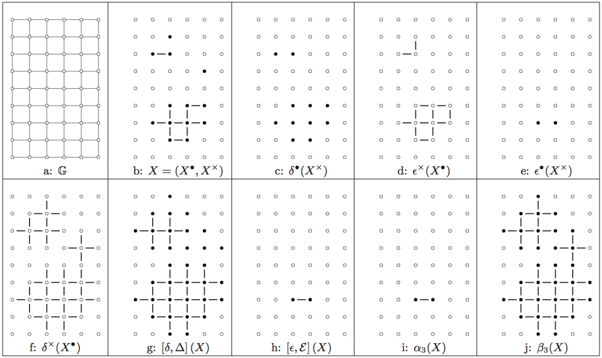

Figure : Graph Dilation And Erosion

$$\epsilon(X^\bullet) = \left\{ x \in \mathbb{G}^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times,\ x, y \in X^\bullet \right\}$$

Edge Erosion

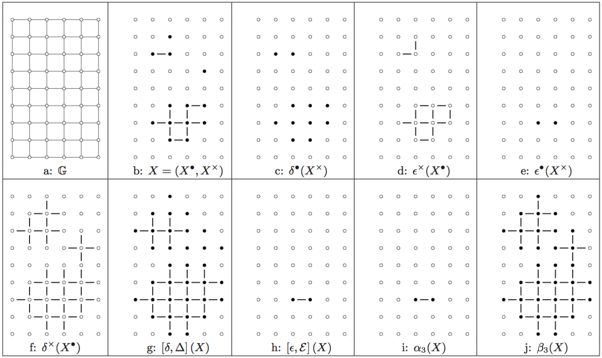

Figure : Graph Dilation And Erosion

$$\mathcal{E}(X^\times) = \left\{ e_{x,y} \in \mathbb{G}^\times \mid \forall e_{x,z}\ e_{y,w} \in \mathbb{G}^\times,\ e_{x,z} \in X^\times,\ e_{y,w} \in X^\times \right\}$$

Graph Opening and Closing

Definition

We denote opening and closing on vertices by γ_1, and ϕ_1, opening and closing on edges by Γ_1, and Φ_1, and opening and closing on graphs by $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$.

1. We define γ_1 and ϕ_1 as $\gamma_1 = \delta \circ \epsilon$ and $\phi_1 = \epsilon \circ \delta$
2. We define Γ_1 and Φ_1 as $\Gamma_1 = \Delta \circ \mathcal{E}$ and $\Phi_1 = \mathcal{E} \circ \Delta$
3. we deine $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$ by $[\gamma, \Gamma]_1 = (\gamma_1(X^\bullet), \Gamma_1(X^\times))$ and $[\phi, \Phi]_1 = (\phi_1(X^\bullet), \Phi_1(X^\times))$.

Graph Half-Opening and Half-Closing

Definition

We denote half-opening and half-closing on vertices by $\gamma_{1/2}$ and $\phi_{1/2}$, half-opening and half-closing on edges by $\Gamma_{1/2}$, and $\Phi_{1/2}$, and half-opening and half-closing on graphs by $[\gamma, \Gamma]_{1/2}$ and $[\phi, \Phi]_{1/2}$.

1. We define $\gamma_{1/2}$ and $\phi_{1/2}$ as $\gamma_{1/2} = \delta^\bullet \circ \epsilon^\times$ and $\phi_{1/2} = \epsilon^\bullet \circ \delta^\times$
2. We define $\Gamma_{1/2}$ and $\Phi_{1/2}$ as $\Gamma_{1/2} = \delta^\times \circ \epsilon^\bullet$ and $\Phi_{1/2} = \epsilon^\times \circ \delta^\bullet$
3. we deine $[\gamma, \Gamma]_{1/2}$ and $[\phi, \Phi]_{1/2}$ by
   $[\gamma, \Gamma]_{1/2} = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$ and
   $[\phi, \Phi]_{1/2} = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$.

Graph Opening and Half-Opening

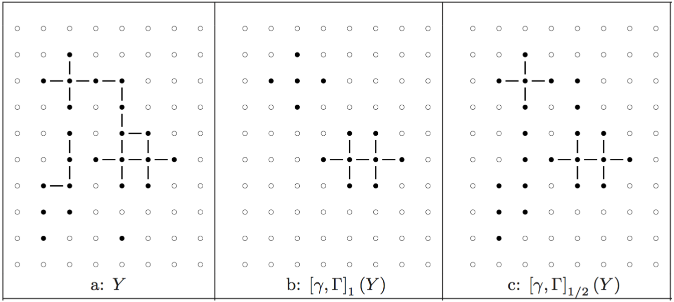

Figure : Graph Opening and Half-Opening

- $\gamma_{1/2}(X^\bullet) = \{x \in X^\bullet \mid \exists e_{x,y} \in \mathbb{G}^\times \ \textit{with}\ y \in X^\bullet\}$
- $\Gamma_{1/2}(Y^\times) = \{u \in G^\times \mid \exists x \in u \ \textit{with}\ \{e_{x,y} \in \mathbb{G}^\times\} \in Y\times\}$

Graph Closing and Half-Closing

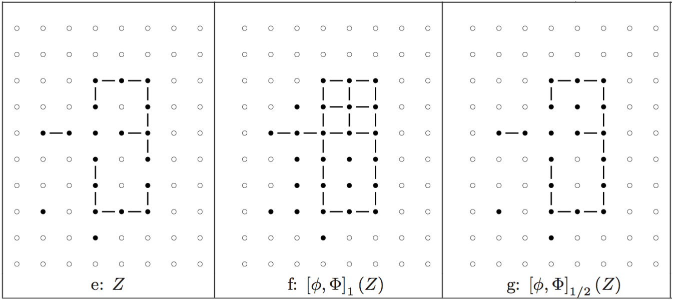

Figure : Graph Closing and Half-Closing

- $\phi_{1/2}(X^\bullet) = \{x \in X^\bullet \mid \forall e_{x,y} \in \mathbb{G}^\times \text{ } \textit{either } x \in X^\bullet \textit{ or } y \in X^\bullet\}$
- $\Phi_{1/2}(Y^\times) = \{e_{x,y} \in \mathbb{G}^\times \mid \exists e_{x,z} \in Y^\times \textit{ and } \exists e_{y,w} \in Y^\times \}$

Granulometries

Definition

We define:

- $[\gamma, \Gamma]_{\lambda/2} = [\delta, \Delta]^i \circ [\gamma, \Gamma]^j_{1/2} \circ [\epsilon, \mathcal{E}]^i$ where $i = \lfloor \lambda/2 \rfloor$ and $j = \lambda - 2 \times \lfloor \lambda/2 \rfloor$
- $[\phi, \Phi]_{\lambda/2} = [\epsilon, \mathcal{E}]^i \circ [\phi, \Phi]^j_{1/2} \circ [\delta, \Delta]^i$, where $i = \lfloor \lambda/2 \rfloor$ and $j = \lambda - 2 \times \lfloor \lambda/2 \rfloor$

Granulometries

Theorem

The families $\left\{[\gamma, \Gamma]_{\lambda/2} \mid \lambda \in \mathbb{B}\right\}$ *and* $\left\{[\phi, \Phi]_{\lambda/2} \mid \lambda \in \mathbb{B}\right\}$ *are granulometries:*

- *for any* $\lambda \in \mathbb{N}$ *,* $[\gamma, \Gamma]_{\lambda/2}$ *is an opening and* $[\phi, \Phi]_{\lambda/2}$ *is a closing.*
- *for any two elements* $\lambda \leq \mu$*, we have* $[\gamma, \Gamma]_{\lambda/2}(X) \supseteq [\gamma, \Gamma]_{\mu/2}$ *and* $[\phi, \Phi]_{\lambda/2} \subseteq [\phi, \Phi]_{\mu/2}$ *where* $\supseteq$ *and* $\subseteq$ *are graph comparisons.*

References

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