MATHEMATICAL MORPHOLOGY IN GEOSCIENCES AND GISci

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II.II. Terrestrial Analysis

Scale invariance and Power-laws in networks

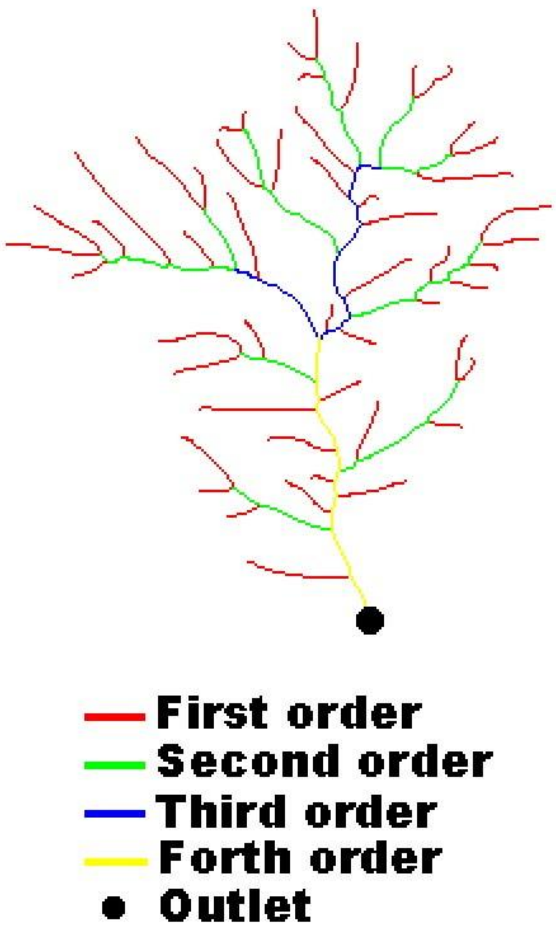
Shape-dependant power-laws

Granulometric analysis

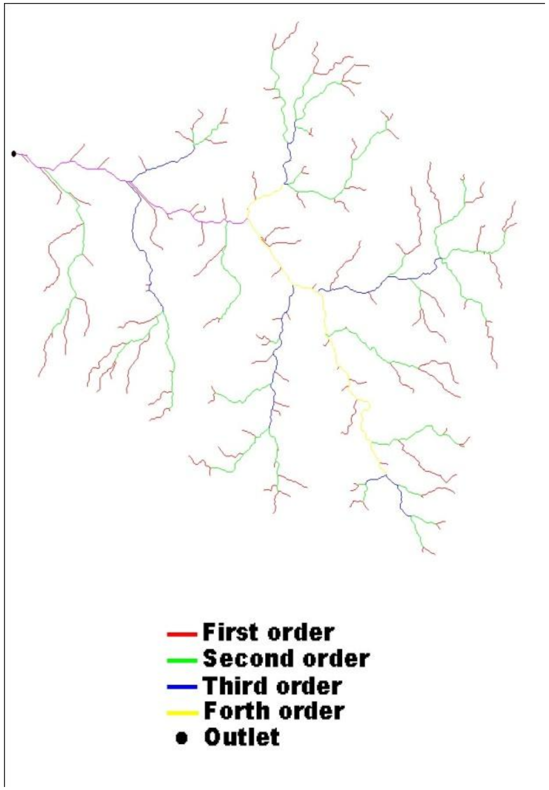
II.II.I. Scale Invariant Power-laws: Morphometry and Allometry of Networks

First step in drainage basin analyses is the classification of stream orders by Horton-Strahler's ordering system (Horton, 1945; Strahler, 1957). The order of the whole tree is defined to be the order of the root. This ordering system has been found to correlate well with important basin properties in a wide range of environments.

This figure shows a sample network classified based on Horton-Strahler's ordering system.



Model network.



Cameron Highland channel network.

Scale Invariant Power-laws: Two Topological Quantities

- Two topological quantities bifurcation ratio (R_b) and length ratio (R_l)

$$R_b = \frac{N_i}{N_{i+1}} \qquad R_l = \frac{L_i}{L_{i-1}}$$

Networks extraction and their properties : Morphometry

- Besides these two ratios, the universal similarity of stream network can be shown through Hack's law and Hurst's law as follows:

- Hack's Law: $L_{mc} \propto A^h$

  where A is the area of basin with main channel length L_{mc}.

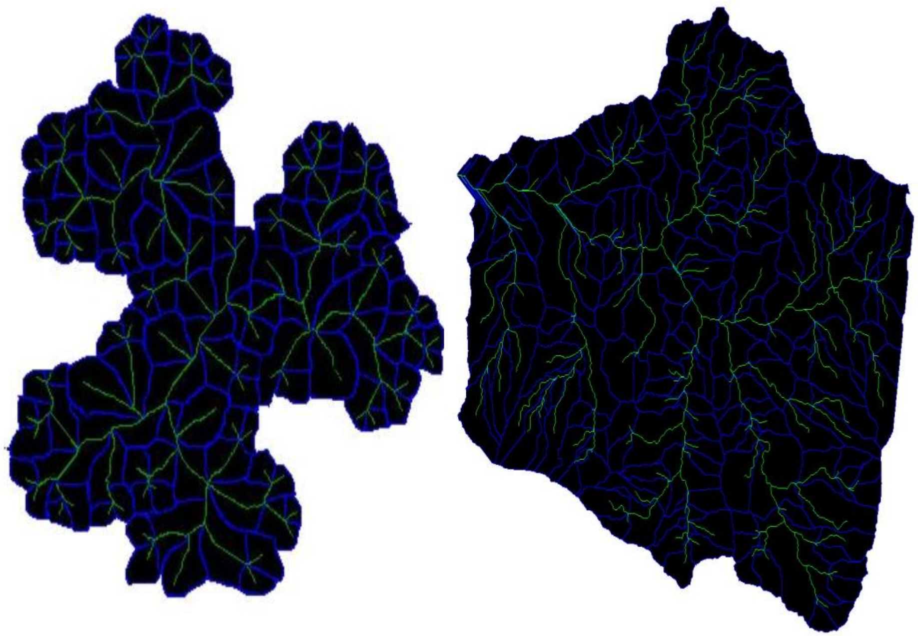
- Hurst's law: $L_{\perp} \propto L_{\parallel}^H$

  where $L_{||}$ is the longitudinal length and L transverse length respectively.

Allometric power-laws

- Allometric power-laws are derived between the basic measures such as basin area, basin perimeter, channel length, longitudinal length and transverse length
- Observed that these power-laws are of universal type as they exhibit similar scaling relationships at all scales.

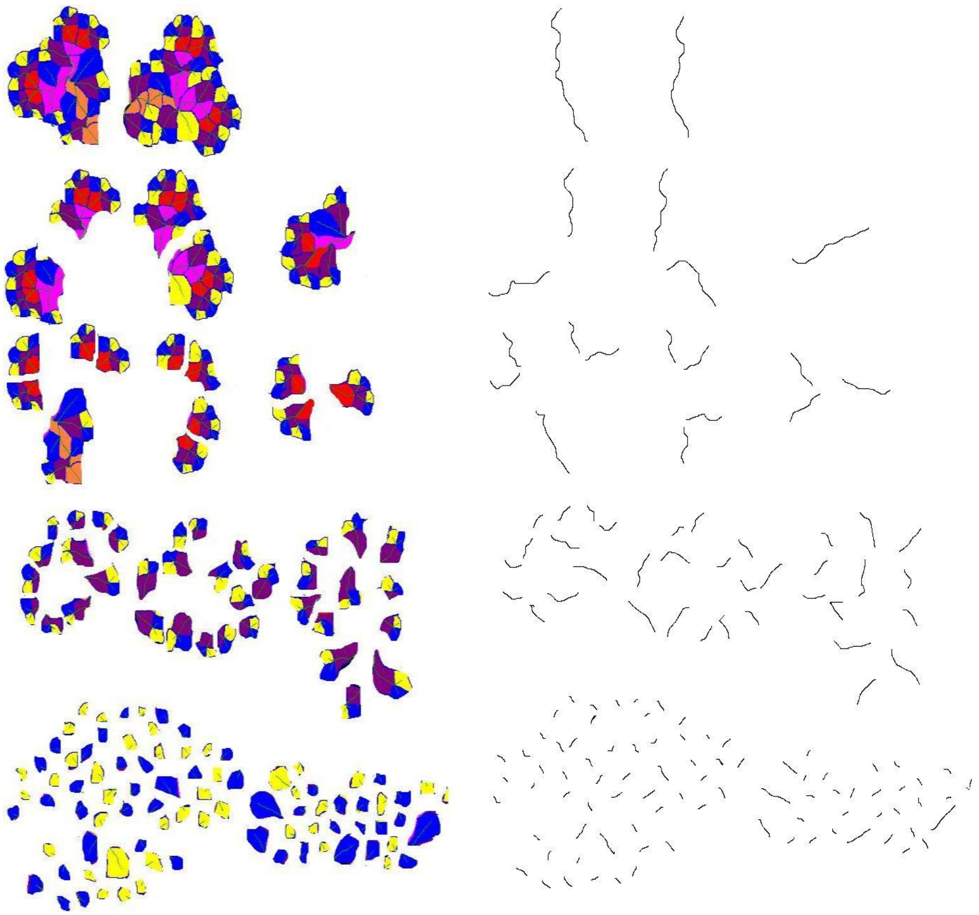
Existing allometric power-laws: Decomposed basins & networks



Existing allometric power-laws : Decomposed basins and networks

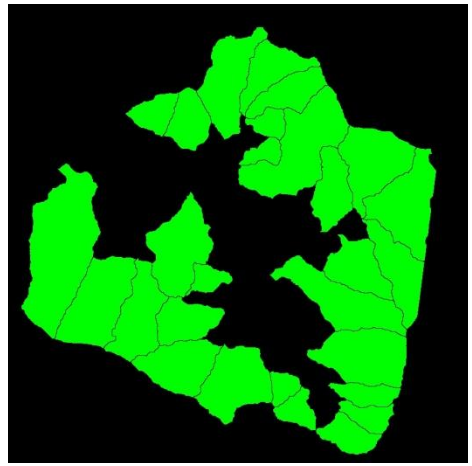
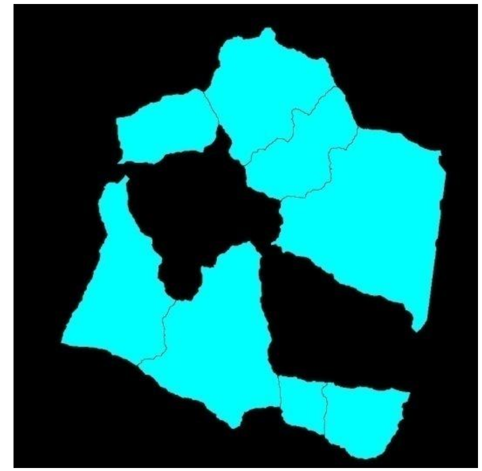
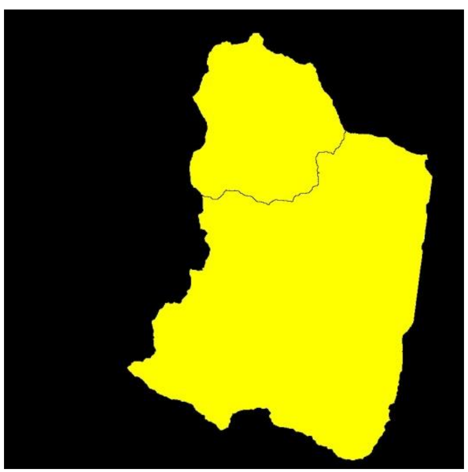
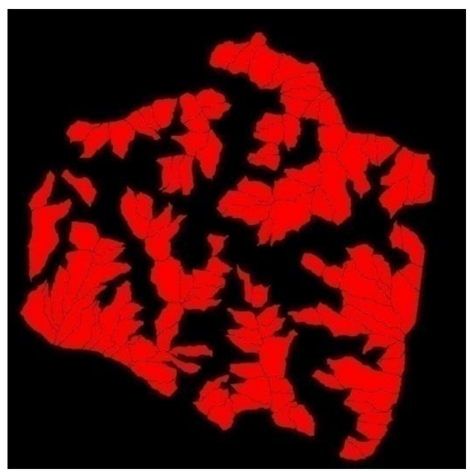
The number of decomposed sub-basins of respective orders from the simulated 6th order F-DEM include:

- two 5th
- five 4th
- ten 3rd
- thirty six 2nd, and
- eighty six 1st order basins.



Existing allometric power-laws :

Decomposed basins and networks

Decomposed sub-basins are

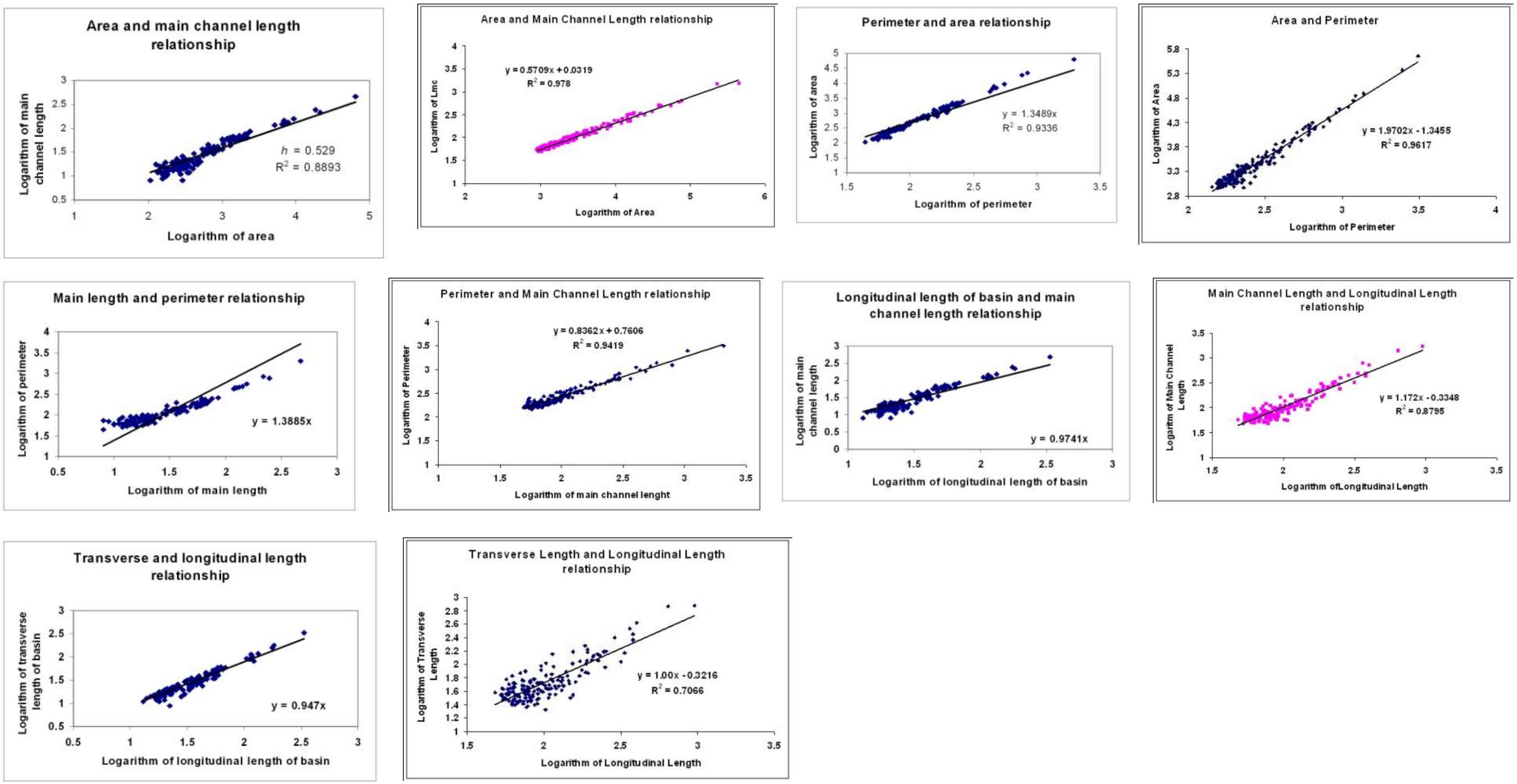
- two 4th
- eight 3rd
- twenty-eight 2nd, and
- one hundred twenty-four 1st

order basins.

Existing allometric power-laws : Basic Measures

Basic measures for a basin, (a) basin area, (b) total channel length, (c) main channel length, (d) basin perimeter, (e) longitudinal length and (f) transverse length.

Scale Invariant allometric power-laws



Allometric relationships among various areal and length parameters for all sub-basins of F-DEM and TOPSAR DEM.

Scale Invariant allometric power-laws

F-DEM TOPSAR DEMs

Relations	Notations	For all orders	Basin's order					
			1	2	3	4	5	6
A and L_{mc}	h	0.53	0.502	0.56	0.56	0.55	0.55	0.56
A and P	α	1.35	1.31	1.36	1.41	1.44	1.48	1.46
P and L_{mc}	β	1.39	1.51	1.32	1.28	1.26	1.23	1.23
L_{mc} and L_{ll}	-	0.97	0.92	1.01	1.04	1.03	0.94	0.95
$L_{\perp}$ and L_{ll}	H	0.95	0.94	0.94	0.96	0.98	0.94	0.98
2h	D_{Lmc}	1.06	1.00	1.11	1.11	1.10	1.10	1.12
$2/\alpha$	D_p	1.48	1.53	1.47	1.42	1.39	1.35	1.37
$1+\frac{D_{Lmc}}{1+H}$	-	1.55	1.52	1.57	1.59	1.56	1.57	1.57

Relations	Notations	For all orders	Basin's order				
			1	2	3	4	5
A and L_{mc}	h	0.57	0.60	0.57	0.50	0.58	0.56
A and P	α	1.97	1.62	1.78	1.78	1.69	1.62
P and L_{mc}	β	0.84	0.78	0.92	0.88	1.09	1.05
L_{mc} and L_{ll}	-	1.17	0.75	1.00	0.92	1.02	1.08
$L_{\perp}$ and L_{ll}	H	1.00	0.39	0.53	0.68	1.00	0.97
2h	D_{Lmc}	1.14	1.20	1.14	1.00	1.16	1.12
$2/\alpha$	D_p	1.02	1.23	1.12	1.12	1.18	1.23
$1+\frac{D_{Lmc}}{1+H}$	-	1.57	1.86	1.74	1.60	1.58	1.57

Existing allometric power-laws : Scaling laws

Our results shown for basins derived from F-DEM and TOPSAR DEM are in good accord with power-laws derived from Optimal Channel Networks (Maritan et. al., 2002) and Random Self-Similar Networks (Veitzer and Gupta 2000) and certain natural river basins.

Novel scaling relationships between travel-time channel networks, convex hulls and convexity measures

Network topology and watershed geometry are important features in terrain characterization.

Travel-time networks are sequence of networks generated by removing the extremities of the network iteratively. Hit-or-Miss transformation and Thinning transformations is used in generating travel-time network. Half-plane closing-based algorithm (Soille, 2005) is employed to generate convex hulls for these travel-time networks.

Length of the travel-time network and area of the corresponding convex hull are used to derive new scaling exponents.

Proposed scaling relationships :

Travel-time networks

- The process of deleting the end points from the networks is named as pruning.
- To decompose the stream network subsets from n = 1 to N, structuring template of B_1 and B_2 are decomposed into various subsets, B_n^i where $i = 1,2,\ldots,8$ and $n = 1,2$

- Both structuring templates are disjointed into eight directions. The intersecting portion of eroded S and eroded Sc by disjointed templates $\{B_1^k\}$ and $\{B_2^k\}$ $k = 1,2,\ldots,8$ respectively are computed to derive pruned version of S.
- The X's in the structuring templates signifies the 'don't care' condition – it doesn't matter whether the pixel in that location has a value of 0 or 1.

$$B_1^1 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad B_1^2 = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad B_1^3 = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad B_1^4 = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$B_1^5 = \begin{matrix} X & 1 & X \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad B_1^6 = \begin{matrix} 0 & 0 & X \\ 0 & 1 & 1 \\ 0 & 0 & X \end{matrix} \quad B_1^7 = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ X & 1 & X \end{matrix} \quad B_1^8 = \begin{matrix} X & 0 & 0 \\ 1 & 1 & 0 \\ X & 0 & 0 \end{matrix}$$

$$B_2^1 = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \quad B_2^2 = \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \quad B_2^3 = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix} \quad B_2^4 = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

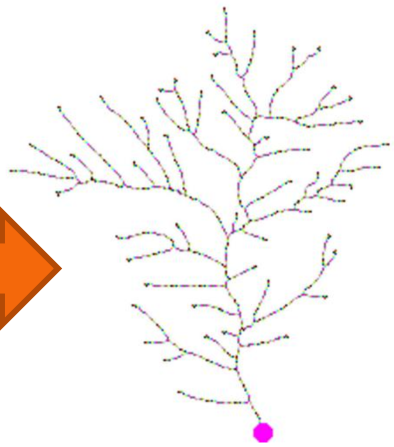
$$B_2^5 = \begin{matrix} X & 0 & X \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \quad B_2^6 = \begin{matrix} 1 & 1 & X \\ 1 & 0 & 0 \\ 1 & 1 & X \end{matrix} \quad B_2^7 = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ X & 0 & X \end{matrix} \quad B_2^8 = \begin{matrix} X & 1 & 1 \\ 0 & 0 & 1 \\ X & 1 & 1 \end{matrix}$$

Proposed scaling relationships : Travel-time networks

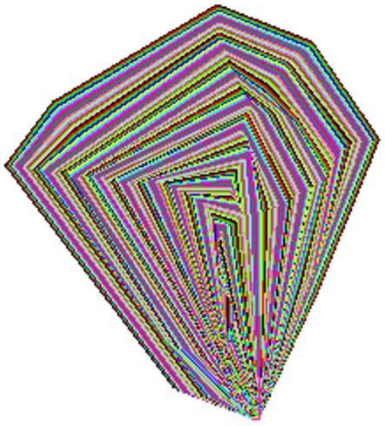
- Mathematically,
  $S * B = (S \ominus B_1^k) \cap (S^c \ominus B_2^k)$ where $B = B_1^k \bigcup B_2^k$
- By subtracting $(S * B)$ from S, a pruned version of S is obtained and expressed as
- $S_1 = S \otimes \{B\}$ where, $S \otimes \{B\} = S - \left(S * B)\right)$
- $\{B\}$ is the sequence of $\left\{(B_1^1, B_1^2, \cdots, B_1^8), (B_2^1, B_2^2, \cdots, B_2^8)\right\}$
- After pruning of S in first pass with B_1, the process continue with pruning with B_2 and so on until S is pruned in the last pass with B_8.
  $S \otimes \{B\} = ((\cdots((S \otimes B^1) \otimes B^2) \cdots) \otimes B^8)$
- The whole process removes the first-encountered open pixels of S and produces S_1.
- Repeating the same process on S_1 will produce S_2 . The process is repeated until no further changes occur, where the closed outlet is reached.

Proposed scaling relationships : Convex hull

Convex hull is the smallest convex set that contains all the points of the network.



Since convex hull represents the basin of network, convex hulls of the travel-time networks are generated.



Proposed scaling relationships : Pruned network and convex hull

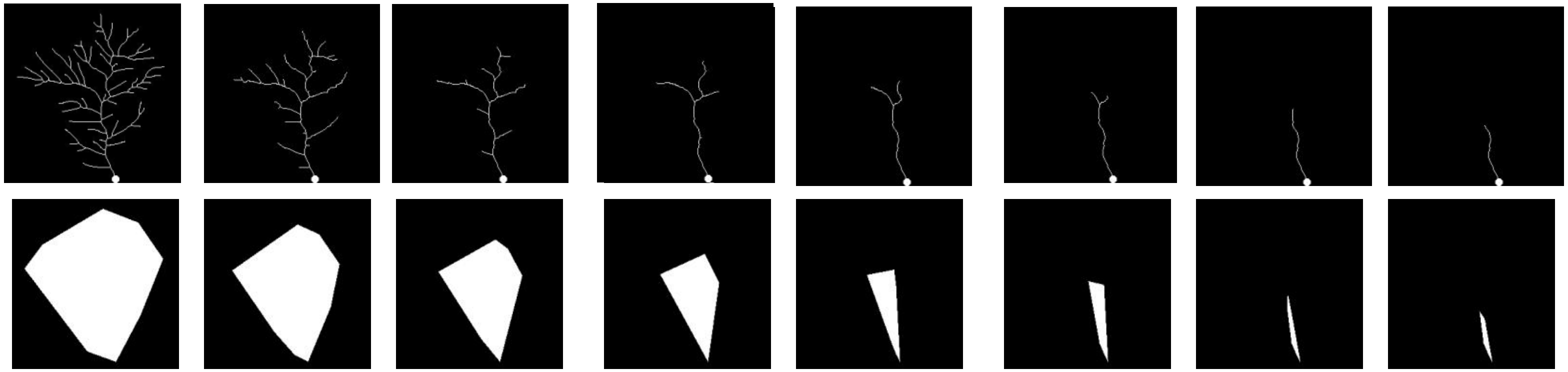
Properties of the pruned network:

1. $S = \bigcup_{n=0}^{N-1} (S_n - S_{n+1})$

2. $S_N \subset S_{N-1} \subset \cdots \subset S_2 \subset S_1 \subset S$

3. $S, S_1, S_2, \cdots, S_N$ obtained by iterative **pruning.**

The final convex polygon containing all the points of S yields C(S).



Proposed scaling relationships

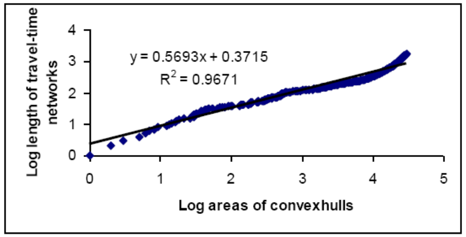
- Network – pruning – network length = S_n
- Convex hull computed – convex hull area = $C(S_n)$
- Convexity measures, CM = ratio between the areas of S_n and $C(S_n)$.

$$L(S_n) \sim A[C(S_n)]^{\alpha}$$

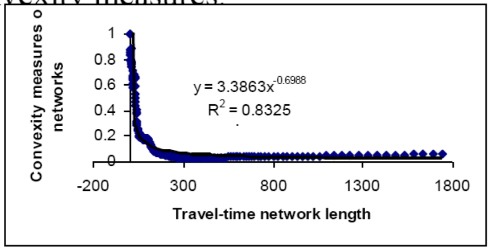
$$CM(S_n) \sim \frac{1}{L(S_n)^{\beta}}$$

$$CM(S_n) \sim \frac{1}{A[C(S_n)]^{\lambda}}$$

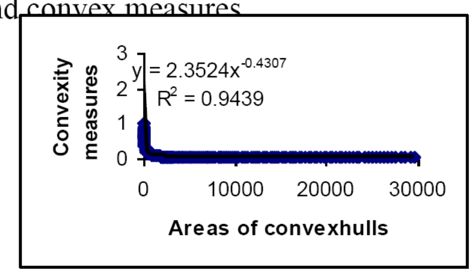
Graph of lengths of the sequential pruned networks versus the corresponding areas of convex hulls.



Relationship between channel lengths and convexity measures.

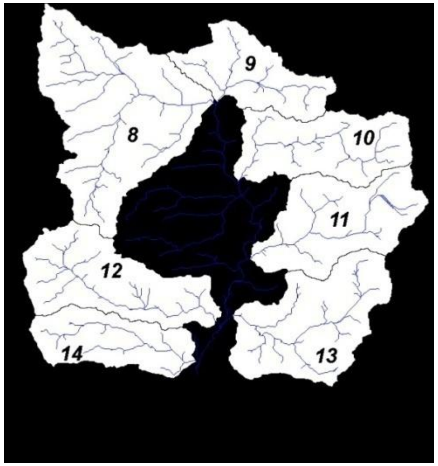


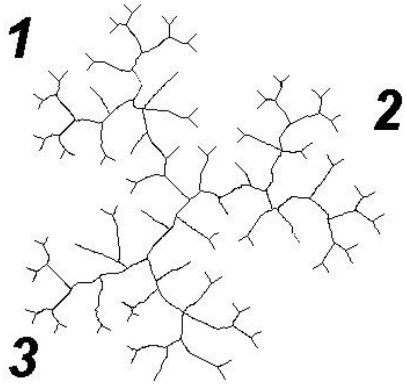
Relationship between areas of convex hulls and convex measures.

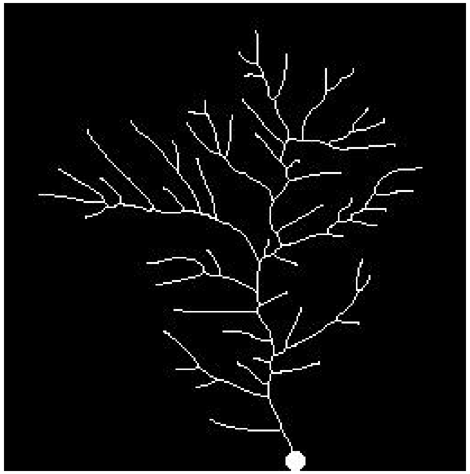


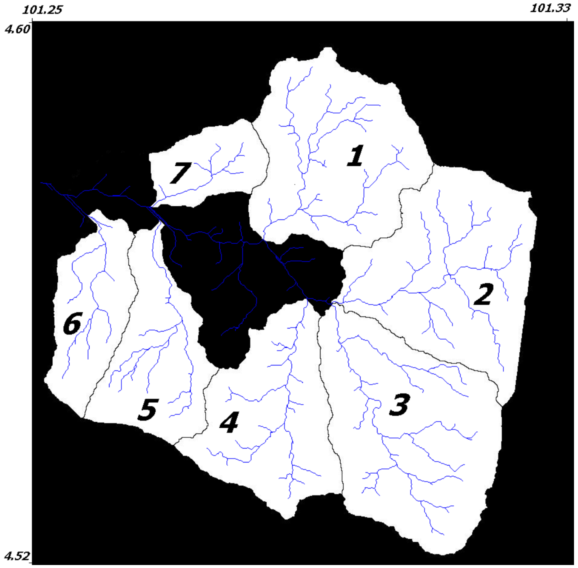
Proposed scaling relationships

- Sample basin
- Simulated F-DEM basins
- Cameron basins
- Petaling basins









Proposed scaling relationships

Network	α, (R^2)	σ, R^2	λ, R^2	R_b	R_l	*h*	*H*
Sample	0.5693, (0.9671)	0.6988, (0.8325)	0.4307, (0.9439)	3.84	1.66	-	-
Basin 1 (Cameron)	0.5777, (0.9883)	0.7109, (0.9358)	0.4223, (0.9783)	3.60	2.21	0.5414	0.9714
Basin 2 (Cameron)	0.5774, (0.9925)	0.7189, (0.9586)	0.4226, (0.9861)	4.35	2.25	0.5561	1
Basin 3 (Cameron)	0.5799, (0.9934)	0.7131, (0.963)	0.4201, (0.9875)	3.31	2.39	0.5612	0.9256
Basin 4 (Cameron)	0.5521, (0.9835)	0.7814, (0.92)	0.4479, (0.9752)	4.47	3.18	0.5671	0.9506
Basin 5 (Cameron)	0.5798, (0.9905)	0.7083, (0.9469)	0.4202, (0.982)	3.31	2.16	0.5766	0.9162
Basin 6 (Cameron)	0.5819, (0.9865)	0.6955, (0.925)	0.4181, (0.9743)	4.00	2.64	0.5746	0.8597
Basin 7 (Cameron)	0.5885, (0.9887)	0.68, (0.9348)	0.4115, (0.9772)	2.82	2.39	0.5548	0.895
Basin 1 (Petaling)	0.5462, (0.969)	0.7741, (0.8561)	0.4538, (0.9557)	5.00	2.57	0.5568	0.9319
Basin 2 (Petaling)	0.5393, (0.9899)	0.8357, (0.9532)	0.4607, (0.9863)	4.00	3.51	0.5828	0.8623
Basin 3 (Petaling)	0.5198, (0.9852)	0.8953, (0.9367)	0.4802, (0.9827)	4.24	3.30	0.597	0.9019
Basin 4 (Petaling)	0.5592, (0.9938)	0.7771, (0.9684)	0.4408, (0.99)	4.24	2.96	0.5807	0.8902
Basin 5 (Petaling)	0.5729, (0.9906)	0.729, (0.9492)	0.4271, (0.9832)	4.79	3.96	0.5844	0.8704
Basin 6 (Petaling)	0.5547, (0.9872)	0.7798, (0.937)	0.4453, (0.9804)	4.89	3.42	0.5713	0.9116
Basin 7 (Petaling)	0.6059, (0.9929)	0.6387, (0.9551)	0.3941, (0.9834)	3.60	3.39	0.5865	0.8312

Allometric power-laws between travel-time channel networks, convex hulls, and convexity measures for model network, networks of Hortonian fractal DEM, and networks of fourteen basins of Cameron Highlands and Petaling region.

Proposed scaling relationships

These proposed scaling exponents are shown for basins derived from simulated F-DEM and TOPSAR DEMs.

These exponents are scale-independent.

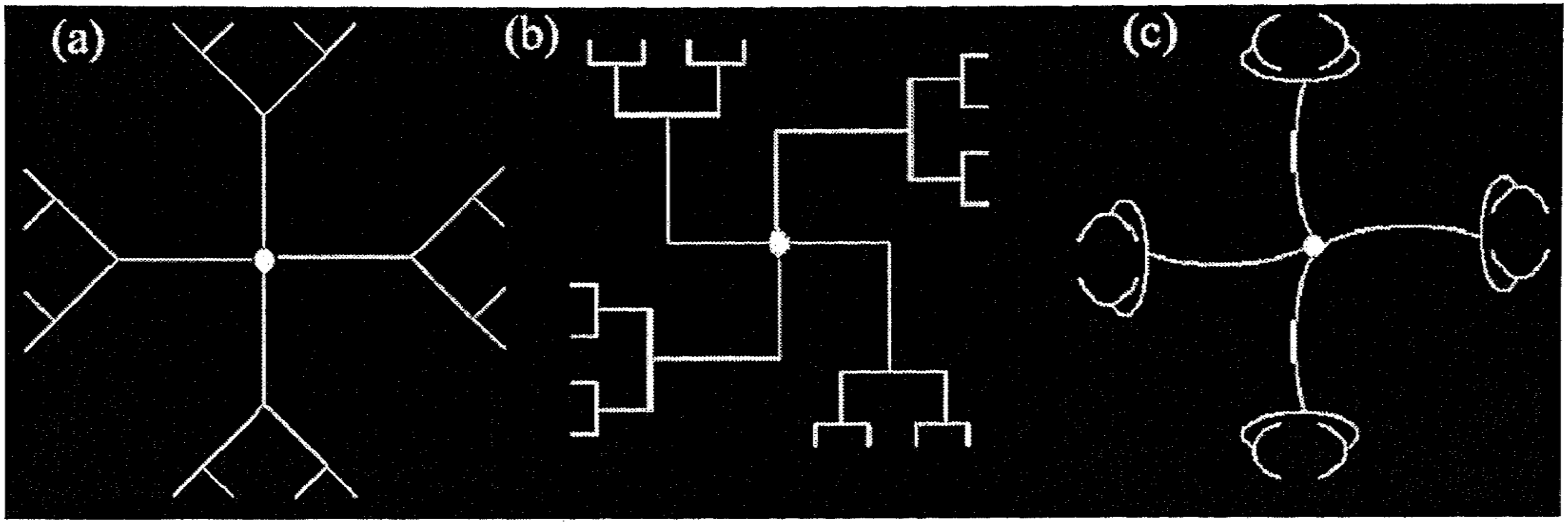
At macroscopic level, these exponents complement with other existing scaling coefficients can be used to identify commonly sharing generic mechanisms in different river basins.

II.II.II. Scale Invariant But Shape Dependent Power-laws

Objectives

To propose morphology based method via fragmentation rules to compute scale invariant but shape-dependent measures of non-network space of a basin.

To make comparisons between morphometry based parameters / dimensions and dimensions derived for non-network space.



Topologically Invariant networks with variant geometric organization

Proposed Technique

Step1: Channel network is traced from topographic map.

Step2: Channel network is dilated and eroded iteratively until the entire basin is filled up with white space. This step is to generate catchment boundary automatically. Dilation followed by erosion is called structural closing, which will smoothen the image.

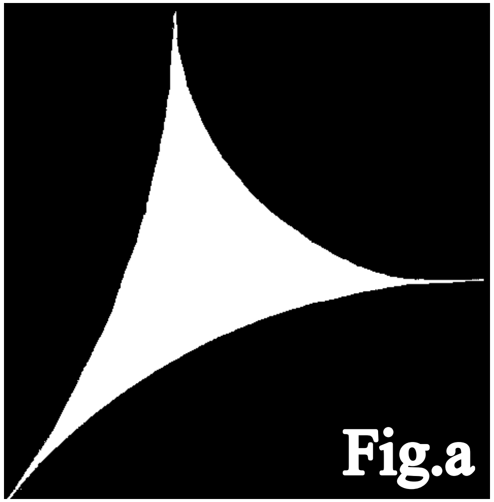
Step3: Generate the basin with channel network and non-network space with boundary by subtracting the channel network from the catchment boundary achieved in Step2.

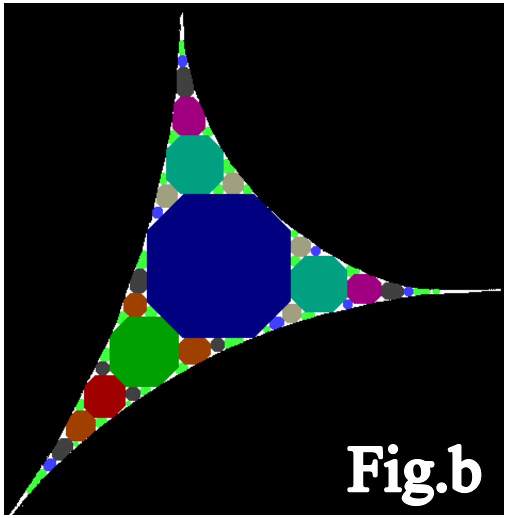
Step4: Structural opening (erosion followed by dilation) is performed recursively in basin achieved in Step3 to fill the entire basin of non-network space with varying size of octagons.

Step5: Assign unique color for each size of octagons.

Step6: Compute morphometry for the basin.

Step7: Compute shape dependent dimension.





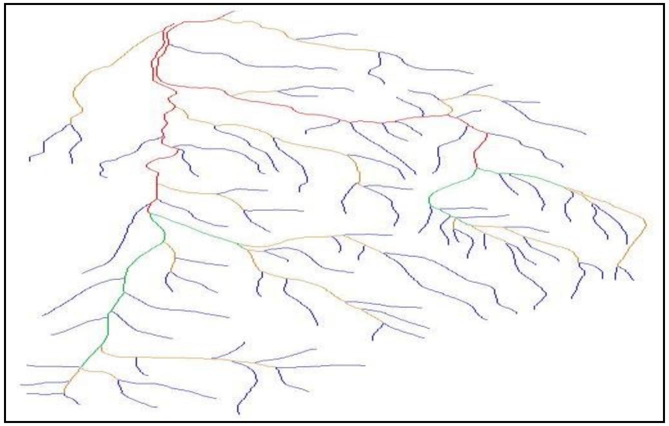
Power law relationship

- As per the previous fig. the slopes of the best-fit lines (α_N and α_A) for number-radius and area-radius relationships yield 2.37 and 1.34.
- These slope values of the best-fit lines provide shape dependent dimensions as $D_N = \alpha_N - 1$ and $D_A = \alpha_A$.
- As in previous Fig., D_N and D_A for non-network space yield 1.37 and 1.34.
- A Power-law relationship is shown in earlier Fig. with an exponent value 1.79 between the area and number of NODs observed with increasing radius of structuring template.

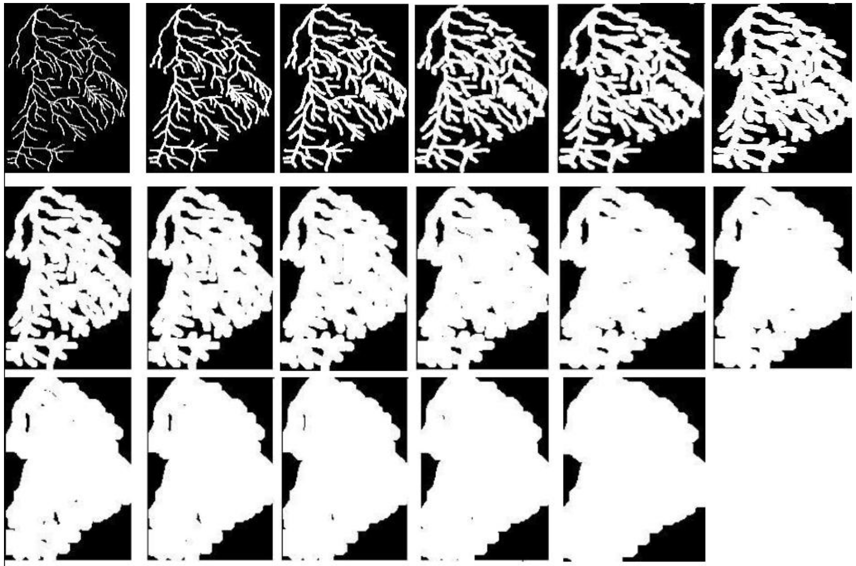
(a) Appollonian Space, and (b) after decomposition by means of octagon.

Algorithm Implementation:

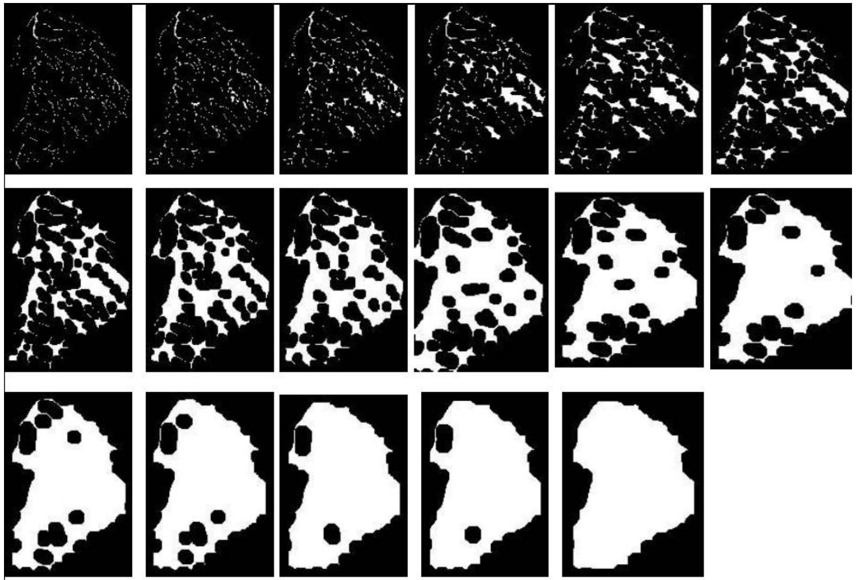
Step 1: **Channel network of sub basin 1**



Step 2: **Close-Hull Generation**

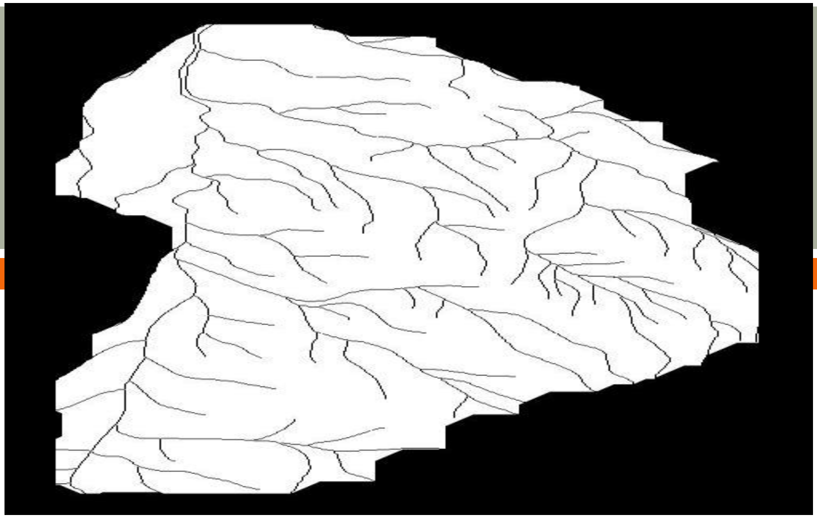


Iterative dilation of channel network of basin 1

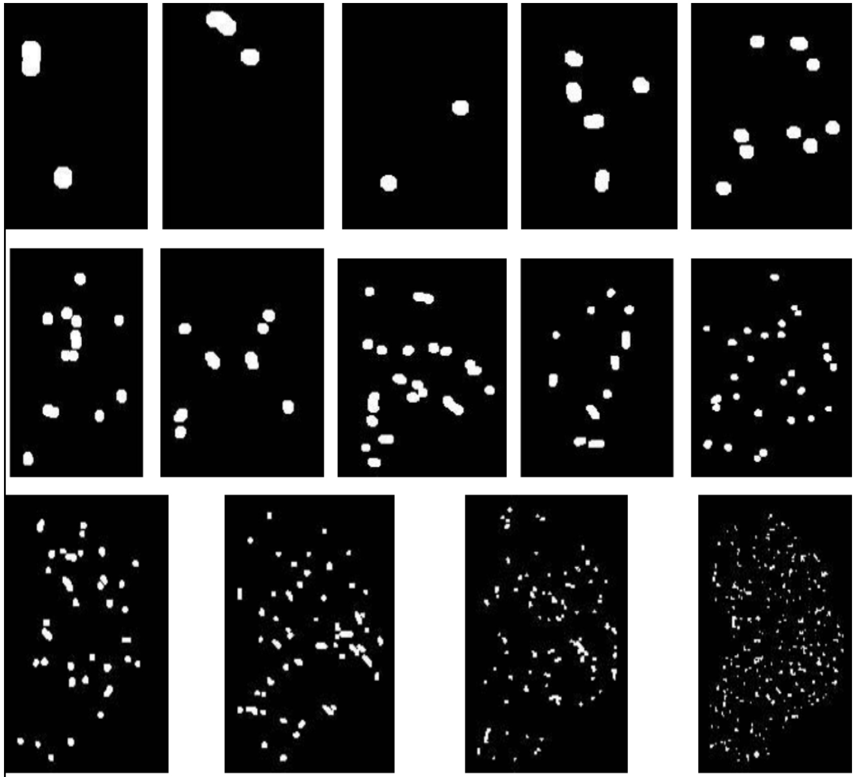


Iterative erosion applied to previous Fig

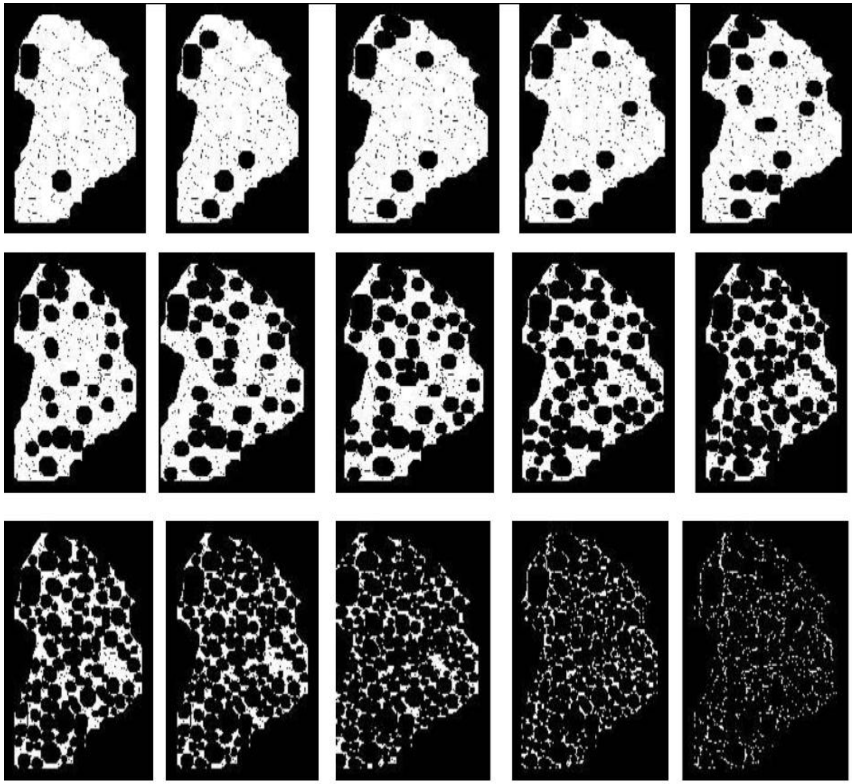
Step 3: Non-network space of basin 1



Iterative erosion applied to step-3 Fig.

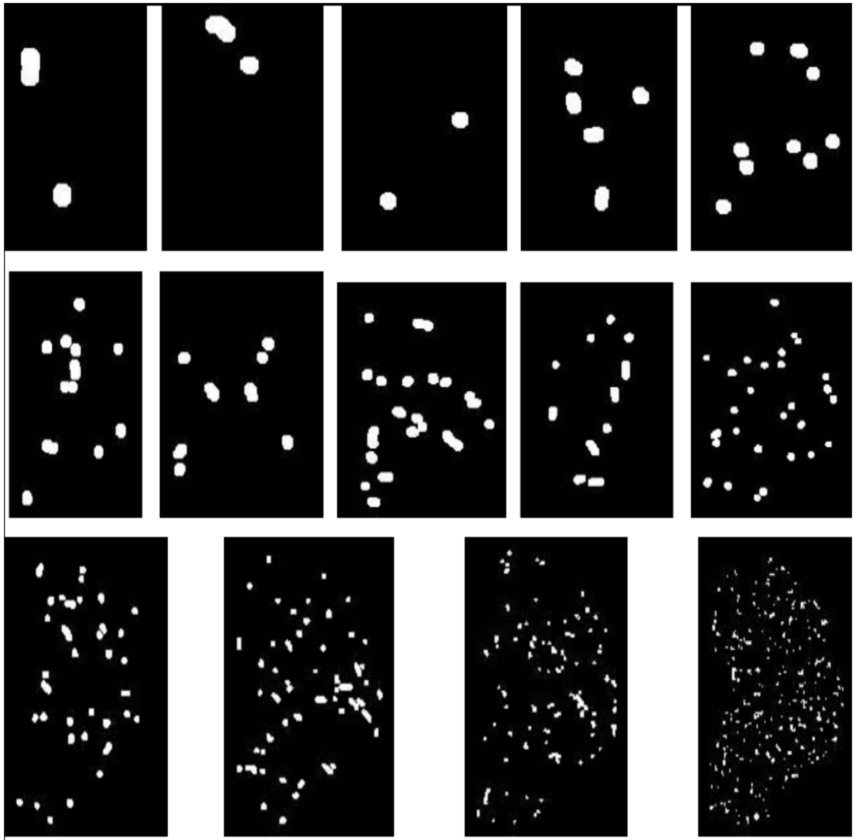


Iterative erosion applied to previous Fig.

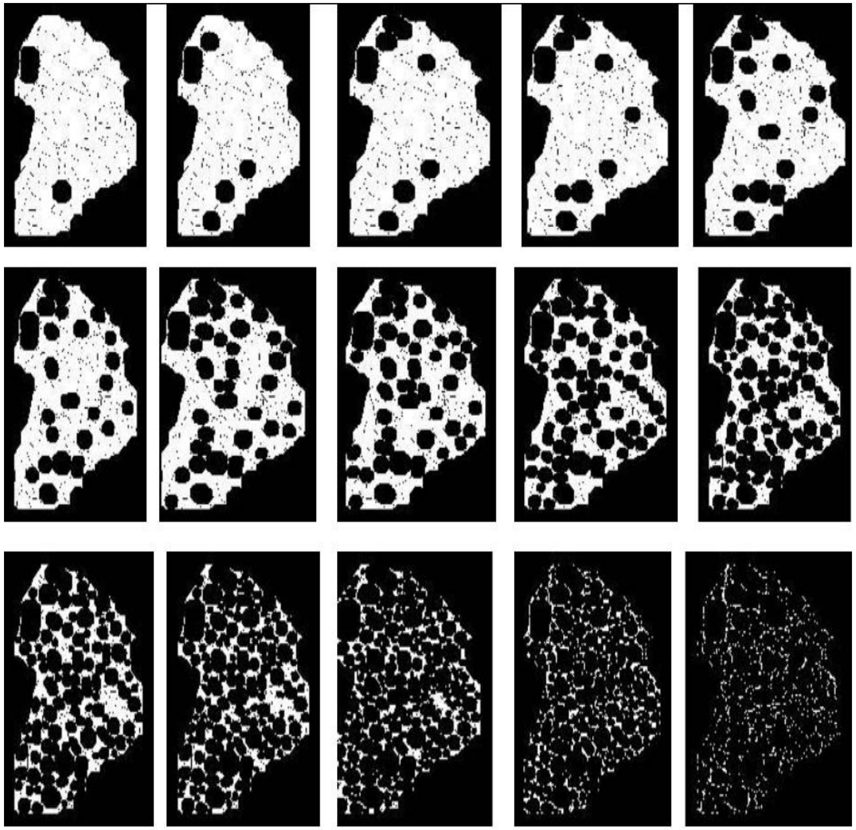


Iterative dilation applied to previous Fig.

Step 4: Non-Network Space Decomposition



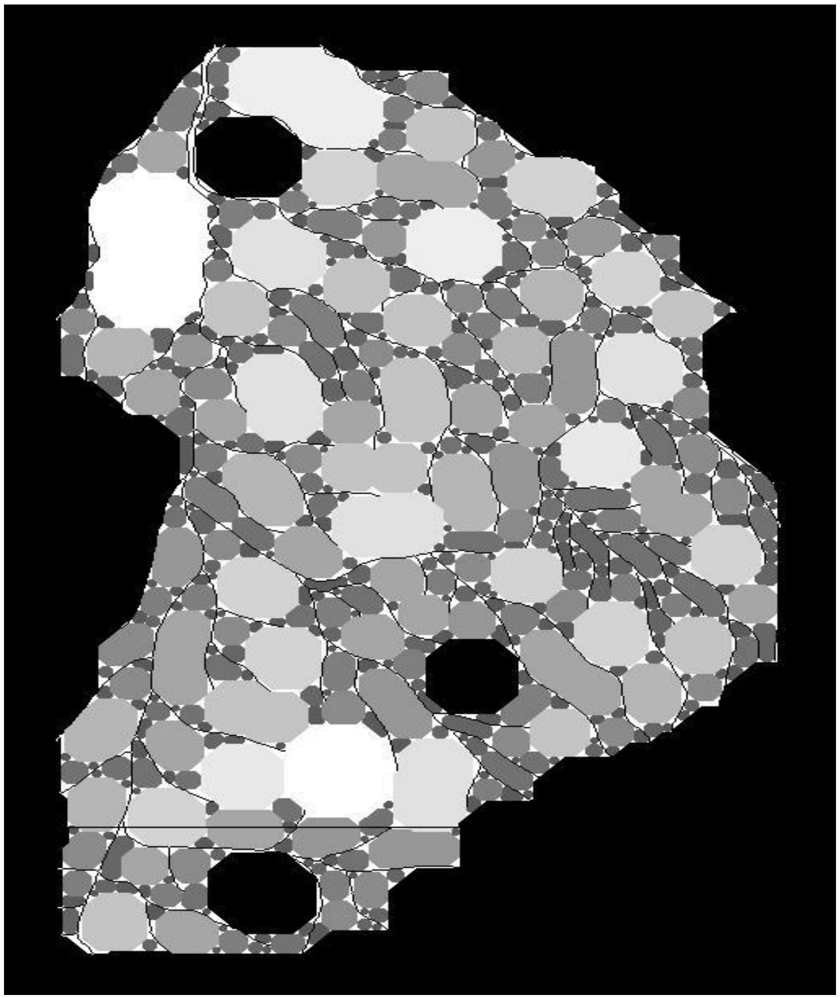
Iterative erosion applied to previous Fig.

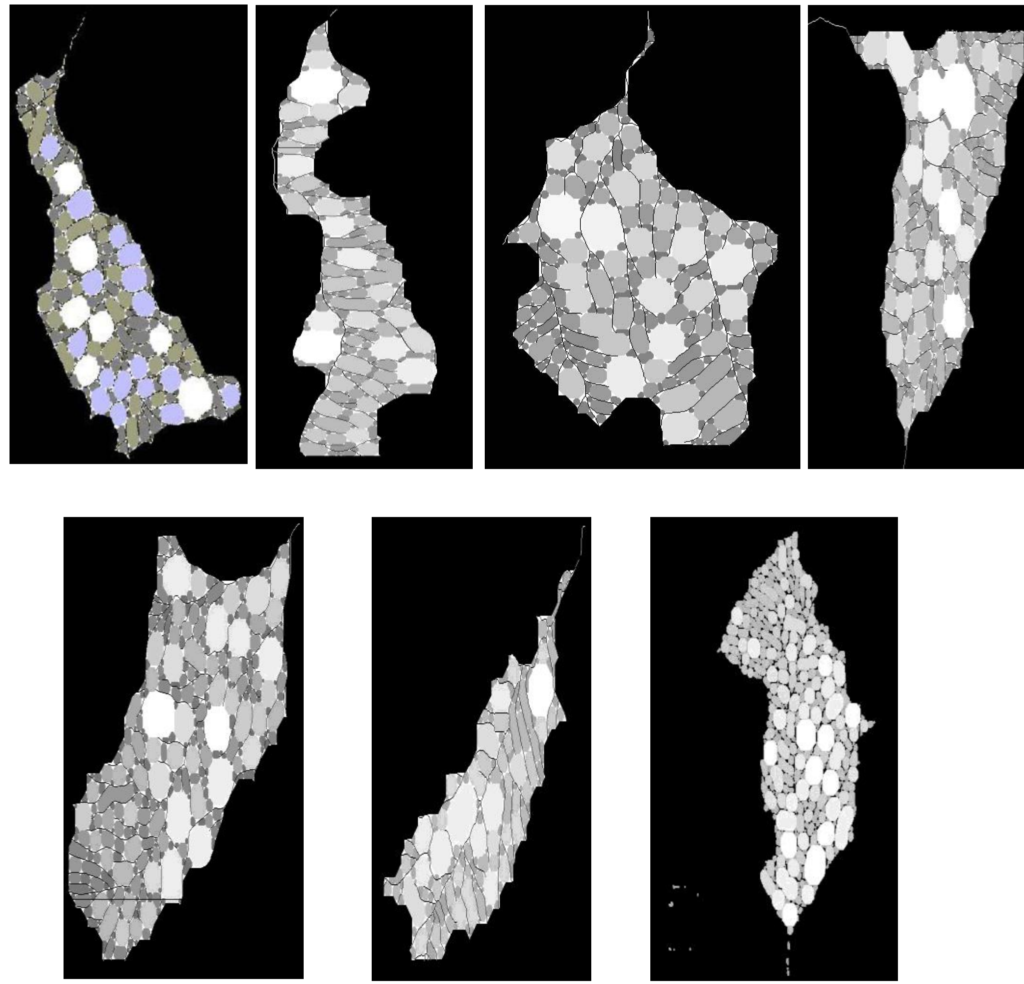


Iterative dilation applied to previous Fig.

Decomposition of Non-network space in to non-overlapping disks of octagon shape of several sizes for basin 1

Non-Network Spaces Packed with Non-Overlapping Disks of basins 2 to 8

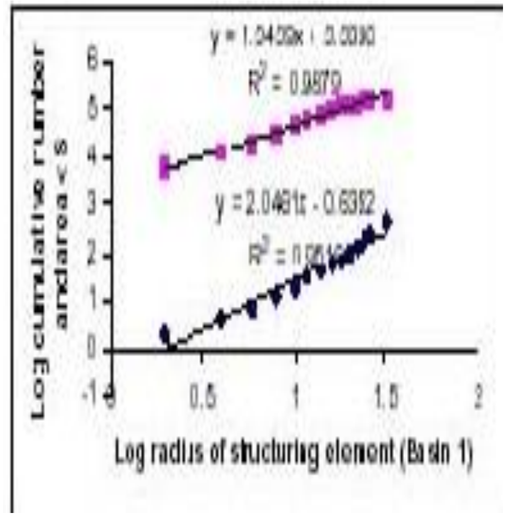


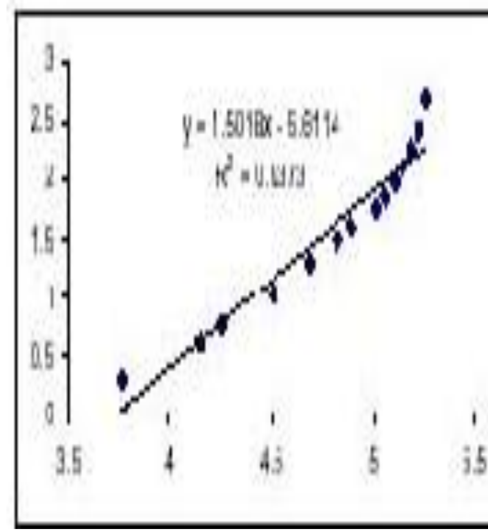


Dimensions derived from morphometry of network and non network space

Morphometric parameter computations achieved through decomposition of non-network space

Basin #	Network FD	Log R_S/ Log R_N	R vs A	R vs N	A vs N
1	1.83	1.93	1.34	2.06	1.50
2	0.86	1.63	1.33	1.23	1.59
3	0.98	1.41	1.02	1.87	1.80
4	2.07	2.01	1.43	2.17	1.52
5	1.73	1.90	1.34	1.94	1.43
6	1.84	2.04	1.13	1.87	1.63
7	1.33	1.61	1.23	2.08	1.70
8	1.65	2.06	1.61	2.38	1.49





Basin number versus varied dimensions derived from morphometry of networks and non-network spaces