On sigma-martingales

Rajeeva L. Karandikar
Director
Chennai Mathematical Institute
rlk@cmi.ac.in
rkarandikar@gmail.com

In order to introduce a sigma-martingale, let us recall some notations and definitions.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a complete probability space and let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration such that $\mathcal{F}_0$ contains all null sets in $\mathcal{F}$. All notions - martingales, adapted, stop times, predictable,... are with respect to this filtration.

Let X be a semimartingale. The class $\mathbb{L}(X)$ of integrands f for which the stochastic integral $\int f\, dX$ can be described as follows.

$\mathbb{L}(X)$ consists of all predictable processes f such that X admits a decomposition $X = M + A$, where M is a locally square integrable martingale and A is a process with finite variation paths with

$$\int_0^t |f_s| d|A|_s < \infty \quad \forall t < \infty \ \ a.s. \tag{1}$$

and there exist stop times $\sigma_k \uparrow \infty$ such that

$$\mathsf{E}[\int_0^{\sigma_k} |f_s|^2 d[M,M]_s] < \infty \quad \forall t < \infty. \tag{2}$$

Then $\int f\,dM$ and $\int f\,dA$ are defined and then

$$\int f\,dX = \int f\,dM + \int f\,dA.$$

One needs to show that this is well defined.

Alternate description of a semimartingale X and the class $\mathbb{L}(X)$ without bringing in the decomposition :

Let X be an r.c.l.l. adapted process. Then $\int f\,dX$ can be defined for all simple predictable processes f directly:

$$f(s) = \sum_{j=0}^{m} a_j 1_{(s_j, s_{j+1}]}(s) \tag{3}$$

where $0 = s_0 < s_1 < s_2 < \ldots < s_{m+1} < \infty$, a_j is bounded $\mathcal{F}_{s_j}$ measurable random variable, $0 \le j \le m$, and

$$\int_0^t f\,dX = \sum_{j=0}^{m} a_j (X_{s_{j+1} \wedge t} - X_{s_j \wedge t}). \tag{4}$$

Let X be an r.c.l.l. adapted process. X is a semimartingale if whenever f^n simple predictable, $|f^n| \leq K$ and $f^n \to 0$ pointwise implies

$$\sup_{0 \leq t \leq T} |\int_0^t f^n dX|$$

converges to zero in probability for all $T < \infty$.
Under this condition, $\int f dX$ can be defined for all bounded predictable processes essentially following steps of Caratheodary extension theorem.

Let X be a semimartingale.

$\mathbb{L}(X)$ consists of predictable process g such that if f^n is a sequence of bounded predictable processes dominated by $|g|$ such that f^n converges to 0 pointwise, then

$$\sup_{0 \le t \le T} \left| \int_0^t f^n dX \right|$$

converges to zero in probability for all $T < \infty$.

Note that if μ is a finite measure on a measuable space, then $\mathbb{L}^1(\mu)$ can be defined as the class of measurable functions g such that if f^n is a sequence of bounded measurable functions dominated by $|g|$ such that f^n converges to 0 pointwise, then $\int f^n d\mu$ converges to zero.

Let M be a martingale. Is $X = \int f \, dM$ a local martingale whenever $f \in \mathbb{L}(M)$?

Answer is No.

Such a process X has been called a sigma-martingale. These were called *Seimimartingales of class* Σ_m by Chou and Emery. The term sigma-martingale is perhaps introduced by Delbaen-Schachermayer.

Sigma martingales can also be described as follows:

A semimartingale X is a sigma-martingale if there exists $(0, \infty)$ valued predictable process ϕ such that $N = \int \phi dX$ is a martingale.

Let M be a martingale and $f \in \mathbb{L}(M)$. When is $X = \int f\, dM$ a local martingale ?

Recall: Burkholder-Davis-Gundy inequality $(p = 1)$: There exist universal constants c^1, c^2 such that for all martingales M with $M_0 = 0$ and for all $t > 0$ one has

$$c^1 \mathrm{E}[([M,M]_T)^{\frac{1}{2}}] \leq \mathrm{E}[\sup_{0 \leq t \leq T} |M_t|] \leq c^2 \mathrm{E}[([M,M]_T)^{\frac{1}{2}}]. \quad (5)$$

As a consequence we have: Let M be a local martingale. Then there exist stop times σ_n increasing to ∞ such that

$$\mathrm{E}[\sqrt{[M,M]_{\sigma_n}}\,] < \infty \ \ \forall n \geq 1 \quad (6)$$

Let M be a local martingale, $f \in \mathbb{L}(X)$ and $N = \int f\, dM$. Since $[N, N] = \int f^2 d[M, M]$.

Using Burkholder-Davis-Gundy inequality, it follows that $X = \int f\, dM$ is a local martingale if and only if there exist stop times $\sigma_k \uparrow \infty$ such that

$$\mathsf{E}[(\int_0^{\sigma_k} |f_s|^2 d[M, M]_s)^{\frac{1}{2}}] < \infty \quad \forall t < \infty.$$

As a consequence, every continuous sigma-martingale is a local martingale.

Also every local martingale N is a sigma-martingale:
let σ_n be bounded stop times increasing to ∞ such that

$$a_n = \mathrm{E}[\sqrt{[N,N]_{\sigma_n}}\,] < \infty.$$

Let ϕ be the predictable process defined by

$$\phi_s = \frac{1}{1+|N_0|}1_{\{0\}}(s) + \sum_{n=1}^{\infty} 2^{-n}\frac{1}{1+a_n}1_{(\sigma_{n-1},\sigma_n]}(s).$$

Then ϕ is $(0,1)$ valued and using Burkholder-Davis-Gundy inequality we can show that $M = \int \phi dN$ is indeed a martingale. Thus N is a sigma-martingale.

Example: A sigma-martingale that is NOT a local martingale:
Let τ be a random variable with exponential distribution (assumed to be $(0,\infty)$-valued without loss of generality) and ξ be a r.v. independent of τ with $\mathrm{P}(\xi = 1) = \mathrm{P}(\xi = -1) = 0.5$. Let

$$M_t = \xi 1_{[\tau,\infty)}(t)$$

and

$$\mathcal{F}_t = \sigma(M_s : s \leq t).$$

For any stop time σ w.r.t. this filtration, it can be checked that either σ is identically equal to 0 or $\sigma \geq (\tau \wedge a)$ for some $a > 0$.

Easy to see that M defined by $M_t = \xi 1_{[\tau,\infty)}(t)$ is a martingale.
Let $f_t = \frac{1}{t} 1_{(0,\infty)}(t)$ and $X_t = \int_0^t f\, dM$.
Then X is a sigma-martingale and

$$[X, X]_t = \frac{1}{\tau^2} 1_{[\tau,\infty)}(t).$$

Thus, for any stop time σ with $\mathrm{P}(\sigma > 0) > 0$, there exists $a > 0$ such that $\sigma \geq (\tau \wedge a)$ and hence

$$\sqrt{[X, X]_\sigma} \geq \frac{1}{\tau} 1_{\{\tau < a\}}.$$

It follows that for any stop time σ, not identically zero, $\mathrm{E}[\sqrt{[X, X]_\sigma}] = \infty$ and so X is not a local martingale.

Why is it called sigma-martingale?
What does it have in common with martingales?

The connection has to do with mathematical finance- the connection between *No Arbitrage* and martingales.

Let $(M_k, \mathcal{F}_k)_{\{0 \leq k \leq n\}}$ be a martingale and let $\{U_k \ : 1 \leq k \leq n\}$ be a bounded predictable process, *i.e.* U_k is $\mathcal{F}_{k-1}$ measurable for $1 \leq k \leq n$. Then

$$N_k = \sum_{i=1}^{k} U_i(M_i - M_{i-1})$$

is a martingale with $N_0 = 0$ and thus

$$\mathrm{P}(N_n \geq 0) = 1 \Rightarrow \mathrm{P}(N_n = 0) = 1.$$

Indeed, even when $(M_k, \mathcal{F}_k)_{\{0 \leq k \leq n\}}$ is not a martingale under P but there exists a equivalent probability measure Q such that $(M_k, \mathcal{F}_k)_{\{0 \leq k \leq n\}}$ is a Q-martingale, it follows that for a bounded predictable process $\{U_k : 1 \leq k \leq n\}$,

$$N_k = \sum_{i=1}^{k} U_i(M_i - M_{i-1})$$

satisfies

$$\mathrm{P}(N_n \geq 0) = 1 \Rightarrow \mathrm{P}(N_n = 0) = 1.$$

We can deduce the same conclusion even if we remove the condition that $\{U_k : 1 \leq k \leq n\}$ are bounded.

The converse to this is also true: if $(M_k, \mathcal{F}_k)_{\{0 \leq k \leq n}$ satisfies for all predictable process for $\{U_k \,:\, 1 \leq k \leq n\}$,

$$N_k = \sum_{i=1}^{k} U_i(M_i - M_{i-1})$$

satisfies

$$\mathrm{P}(N_n \geq 0) = 1 \Rightarrow \mathrm{P}(N_n = 0) = 1$$

then there exists there exists a equivalent probability measure Q such that $(M_k, \mathcal{F}_k)_{\{0 \leq k \leq n}$ is a Q-martingale. This result was proven only in 1990 by Dalang-Morton-Willinger though this connection was actively pursued since late '70s.

If M_k denotes the discounted stock price on day k, $\{U_j : j \geq 1\}$ represents a trading strategy and

$$N_k = \sum_{i=1}^{k} U_i(M_i - M_{i-1})$$

represents gain from the strategy. If $\mathrm{P}(N_n \geq 0) = 1$ and $\mathrm{P}(N_n > 0) > 0$, then $\{U_j : j \geq 1\}$ is called an arbitrage opportunity.

The DMZ theorem can be recast as *No Arbitrage* (NA) if and only if *Equivalent Martingale Measure* (EMM) exists.

When one considers infinite time horizon or trading in continuous time, one not only has to rule out arbitrage opportunities, but also approximate arbitrage opportunities.

We consider some examples that illustrate these points.

Let $\{S_n : n \geq 1\}$ be a process (model for price of a stock). Let

$$\mathcal{C} = \{V : \exists n, \text{ predictable } U_1, U_2, \ldots, U_n \text{ s.t. } V \leq \sum_{j=1}^{n} U_j(S_j - S_{j-1})\}.$$

The NA condition can be equivalently written as:

$$\mathcal{C} \cap \mathbb{L}_+^\infty = \{0\}.$$

Let $\{Z_n : n \geq 1\}$ be independent random variables on $(\Omega, \mathcal{F}, \mathrm{P})$ with

$$\mathrm{P}(Z_n = \frac{1}{2}) = p, \quad \mathrm{P}(Z_n = -\frac{1}{2}) = q$$

where $0 < q < p < 1$ and $p + q = 1$. Let $S_n = \prod_{j=1}^{n}(1 + Z_j)$. There is exactly one probability measure Q on $\sigma(S_n : n \geq 1)$ such that $\{S_n\}$ is a martingale, namely the one under which $\{Z_n : n \geq 1\}$ are i.i.d. with $\mathrm{P}(Z_n = \frac{1}{2}) = \frac{1}{2}$ and $\mathrm{P}(Z_n = -\frac{1}{2}) = \frac{1}{2}$. By Kakutani's theorem, Q and P are orthogonal. Thus there is no EMM. However, for every finite horizon there is such a measure and as a result, NA holds.

For infinite horizon or in continuous time, we need to rule out *approximate arbitrage i.e.* a sequence $Z_n \in \mathcal{C}$ such that Z_n converges to $Z \in \mathbb{L}^{\infty}_{+}$ implies $Z = 0$.

But this needs to be defined carefully. The classic strategy of betting on a sequence of fair coin tosses that doubles the investment needs to be ruled out as it is clearly an arbitrage opportunity though the underlying process is a martingale.

An admissible trading strategy is a predictable sequence $\{U_k\}$ such that $\exists K < \infty$ with

$$\mathrm{P}(\sum_{j=1}^{m} U_j(S_j - S_{j-1}) \geq -K) = 1 \ \ \forall m \geq 1.$$

Here K is to be interpreted as credit limit of the investor using the trading strategy.
Let $\mathcal{A}$ denote class of admissible strategies.

Now the class of attainable positions $\mathcal{K}$ is defined as

$$\mathcal{K} = \{W : \exists n \geq 1, \{U_k\} \in \mathcal{A} \text{ s.t. } W = \sum_{j=1}^{n} U_j(S_j - S_{j-1})\}.$$

Let

$$\mathcal{C} = \{V \in\in \mathbb{L}^\infty \ : \ \exists W \in \mathcal{K} \text{ with } V \leq W\}.$$

The *No Approximate Arbitrage* (NAA) condition is:

$$\bar{\mathcal{C}} \cap \mathbb{L}_+^\infty = \{0\}$$

where $\bar{\mathcal{C}}$ is the closure in $\mathbb{L}^\infty$. Now it can be shown that NAA holds for $\{S_k\}$ if and only if there exists a equivalent local martingale measure (ELMM).

Another interesting observation: Every bounded mean zero random variable is attainable (called completeness of market) if and only if ELMM is unique.

Thus we have : NAA if and only if ELMM exists and market is complete if and only if ELMM is unique.

Let us move to continuous time. Let $\{S_t : 0 \leq t \leq T\}$ denote the stock price. Let $\mathcal{F}_t$ be the σ field generated by all observables up to time t. A simple trading strategy is a predictable process f given by

$$f(s) = \sum_{j=0}^{m} a_j 1_{(s_j, s_{j+1}]}(s) \tag{7}$$

where $0 = s_0 < s_1 < s_2 < \ldots < s_{m+1} < \infty$, a_j is bounded $\mathcal{F}_{s_j}$ measurable random variable, $0 \leq j \leq m$. For such a trading strategy f, the value of the holding is given by

$$V(f)_t = \sum_{j=0}^{m} a_j (X_{s_{j+1} \wedge t} - X_{s_j \wedge t}). \tag{8}$$

Such a simple trading strategy is said to be admissible if for some constant K

$$\mathrm{P}(V(f)_t \geq -K) = 1.$$

K represents the credit limit of the investor. Let $\mathcal{A}_s$ represent class of admissible simple strategies.

Let

$$\mathcal{C}_s = \{Y \in \mathbb{L}^\infty : \exists f \in \mathcal{A}_s \text{ s.t. } Y \leq V(f)_T\}.$$

We will say that NAA holds in the class of simple trading strategies if

$$\bar{\mathcal{C}}_s \cap \mathbb{L}^\infty_+ = \{0\}$$

where $\bar{\mathcal{C}}$ is the closure in $\mathbb{L}^\infty$.
Delbaen-Schachermayer have shown that if NAA holds in the class of simple trading strategies for an r.c.l.l. process $\{S_t : 0 \leq t \leq T\}$ then S is a semimartingale.

Suppose the model for stock price S is a semimartingale. An admissible trading strategy is a predictable process $f \in \mathbb{L}(S)$ such that for some constant K $\mathsf{P}(\int_0^t f\,dS \geq -K) = 1$. Let $\mathcal{A}$ denote the class of admissible trading strategies.

Let the class of attainable claims be defined by

$$\mathcal{K} = \{W : \exists f \in \mathcal{A}, a \geq 0 \text{ s.t. } W = \int_0^T f\,dS\}$$

and

$$\mathcal{C} = \{Y \in \mathbb{L}^\infty : \exists W \in \mathcal{K}\ s.t. Y \leq W\}.$$

Delbaen-Schachermayer showed in 1994 that if S is locally bounded, then

$$\bar{\mathcal{C}} \cap \mathbb{L}^{\infty}_{+} = \{0\}$$

if and only if ELMM exists.

The property $\bar{\mathcal{C}} \cap \mathbb{L}^{\infty}_{+} = \{0\}$ has been called *No Free Lunch with Vanishing Risk* (NFLVR) by them.

In this case too one has the result that market is complete if and only if ELMM is unique. The completeness of martingale here is same as *all bounded mean zero random variables admit stochastic integral representation w.r.t.* S.

Delbaen-Schachermayer showed in 1998 that more generally NFLVR holds *i.e.*

$$\bar{\mathcal{C}} \cap \mathbb{L}^{\infty}_{+} = \{0\}$$

if and only if there exists an equivalent probability measure Q under which S is a sigma-martingale (ESMM exists).

Here again we have that market is complete if and only if ESMM is unique.

For simplicity of notations, we have only described results in the 1-dimensional case. But equivalence of NFLVR and existence of ELMM / ESMM as well as completeness of markets and uniqueness of ELMM / ESMM holds when we have a d-stocks.