Recent results on variants of random geometric graphs

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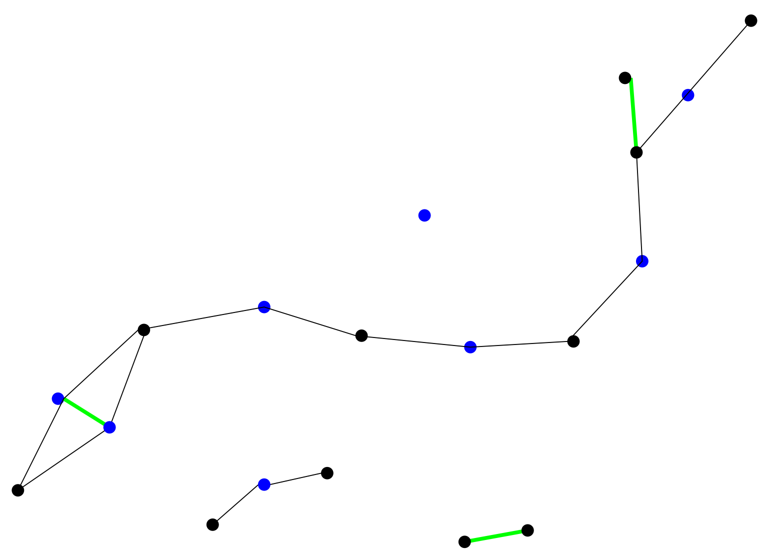
Geometric graphs

Let $d \in \mathbb{N}$ with $d \geq 2$. Let $r > 0$. Given disjoint, locally finite $\mathcal{X} \subset \mathbb{R}^d$, $\mathcal{Y} \subset \mathbb{R}^d$, define the *geometric graph* $G(\mathcal{X}, r)$ ($G = (V, E)$) by

$$V = \mathcal{X}, E = \{\{x, x'\} : |x - x'| \leq r\}$$

and the *bipartite geometric graph* $G(\mathcal{X}, \mathcal{Y}, r)$ by

$$V = \mathcal{X} \cup \mathcal{Y}, E = \{\{x, y\} : x \in \mathcal{X}, y \in \mathcal{Y}, |x - y| \leq r\}.$$



Random geometric graphs

Given $\lambda, \mu > 0$, let $\mathcal{P}_\lambda$ and $\mathcal{Q}_\mu$ be independent homogeneous Poisson point processes of intensity λ, μ resp. in $\mathbb{R}^d$. Let $\mathcal{I}$ be the class of graphs which *percolate*, i.e. have an infinite component. By a standard zero-one law, given also $r > 0$ we have

$$\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] \in \{0, 1\};$$

$$\mathbb{P}[G(\mathcal{P}_\lambda, r) \in \mathcal{I}] \in \{0, 1\}.$$

The graph $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$ is a (loose) continuum analogue to AB percolation on a lattice (e.g. Halley (1980), Appel and Wierman (1987)), where each vertex is either type A or type B, and one is interested in infinite alternating paths.

Critical values. Given $\lambda > 0$ and $r > 0$, define

$$\mu_c(r,\lambda) := \inf\{\mu : \mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}] = 1\}$$

with $\inf\{\} := +\infty$. Set

$$\lambda_c^{AB}(r) := \inf\{\lambda : \mu_c(r,\lambda) < \infty\};$$

and

$$\lambda_c(r) := \inf\{\lambda : \mathbb{P}[G(\mathcal{P}_\lambda, r) \in \mathcal{I}] = 1\}.$$

THEOREM 1 (Iyer and Yogeshwaran (2012), Penrose (2014)):

$$\lambda_c^{AB}(r) = \lambda_c(2r)$$

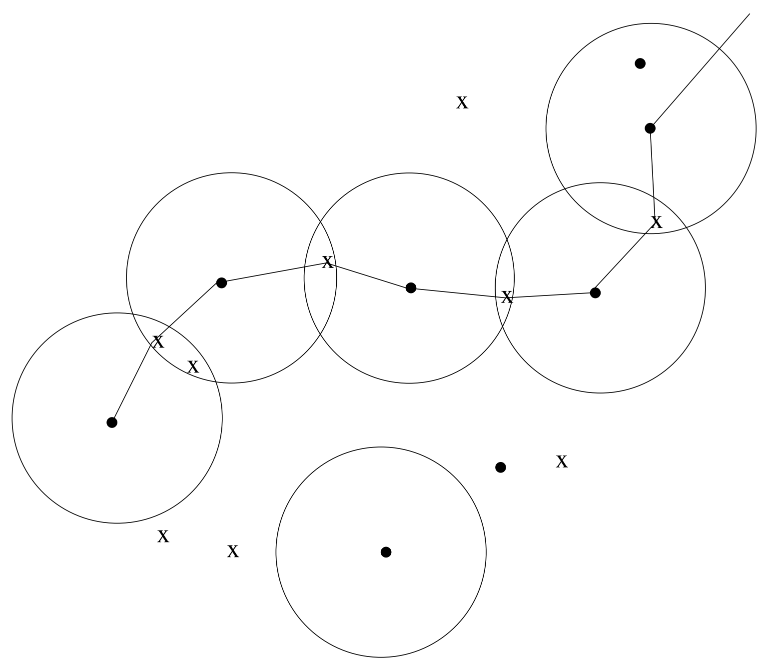
and

$$\mu_c(r, \lambda_c(2r) + \delta) = O(\delta^{-2d}|\log \delta|) \text{ as } \delta \downarrow 0.$$

Proving $\lambda_c^{AB}(r) \geq \lambda_c(2r)$ is trivial

If $\lambda > \lambda_c^{AB}(r)$ then $\exists \mu$ with $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$ a.s..

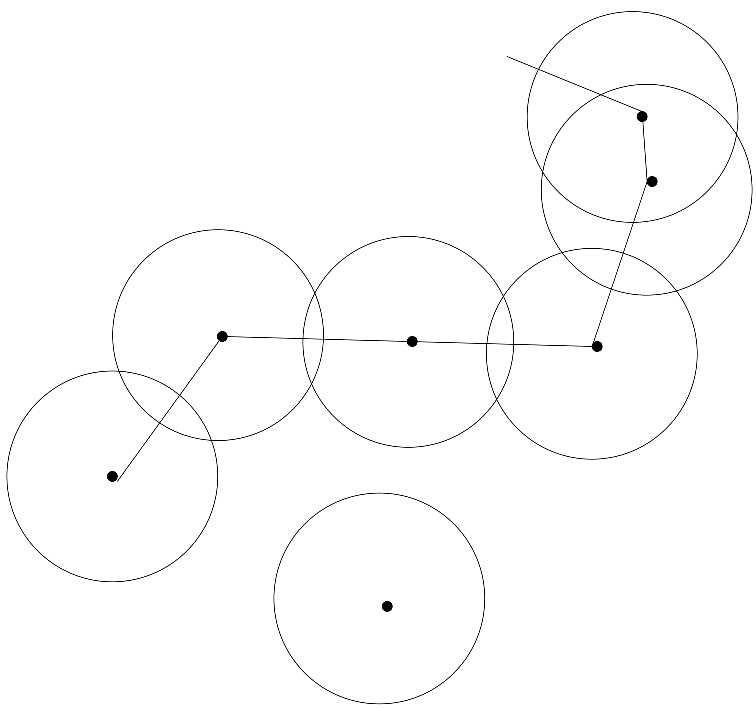
Then also $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$ a.s., so $\lambda \geq \lambda_c(2r)$.



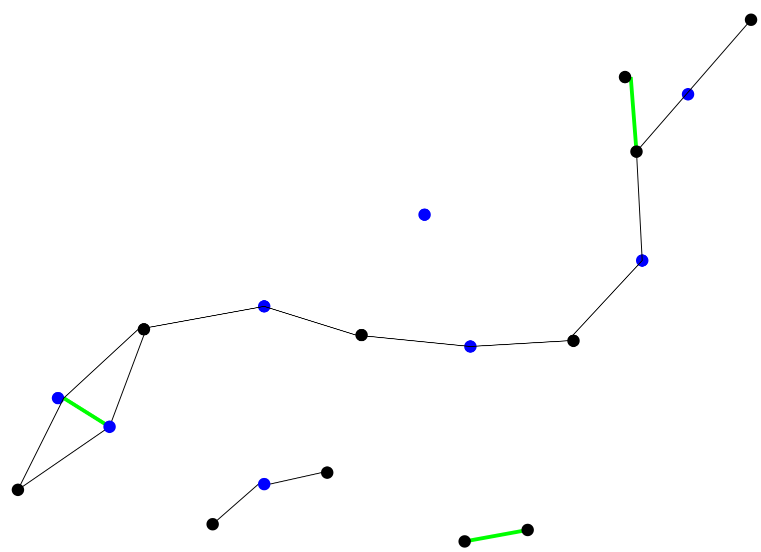
Proving $\lambda_c^{AB}(r) \leq \lambda_c(2r)$ is less trivial

Suppose $\lambda > \lambda_c(2r)$, so $G(\mathcal{P}_\lambda, 2r) \in \mathcal{I}$ a.s. We want to show:

$\exists \mu$ (large) such that $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r) \in \mathcal{I}$ a.s., so $\lambda \geq \lambda_c^{AB}(r)$.



Discretization of $G(\mathcal{P}_\lambda, \mathcal{Q}_\mu, r)$**.** Divide $\mathbb{R}^d$ into cubes of side ε (small). Say each cube C is A-occupied if $\mathcal{P}_\lambda(C) > 0$ is and is B-occupied if $\mathcal{Q}_\mu(C) > 0$. Induces bipartite site-percolation on ε-grid.



Sketch proof of $\lambda_c^{AB}(r) \leq \lambda_c(2r)$ (1): Discretization Suppose $\lambda > \lambda_c(2r)$. Then $\exists\ s < r$ and $\nu < \lambda$ with $G(\mathcal{P}_\nu, 2s) \in \mathcal{I}$ a.s.

For $\varepsilon > 0$, $p, q \in [0,1]$; let $\mathbb{P}_{p,q,\varepsilon}$ be the measure under which each site $z \in \varepsilon\mathbb{Z}^d$ is A-occupied with probability p and (independently) B-occupied with probability q (it could be both, or neither). Let $\mathcal{A}$ be the set of A-occupied sites and $\mathcal{B}$ the set of B-occupied sites. Set $t = (r+s)/2$ and $\varepsilon = (r-t)/(9d)$. Can show

$$\mathbb{P}_{p_\nu,1,\varepsilon}[G(\mathcal{A},\mathcal{B},t) \in \mathcal{I}] = 1$$

where $p_\nu = 1 - \exp(-\nu\varepsilon^d)$ (Prob that ε-cube has ≥ 1 point of $\mathcal{P}_\nu$).

Next lemma will show $\exists q < 1$:

$$\mathbb{P}_{p_\lambda,q,\varepsilon}[G(\mathcal{A},\mathcal{B},t) \in \mathcal{I}] = 1,$$

which implies $\mathbb{P}[G(\mathcal{P}_\lambda, \mathcal{P}_\mu, r) \in \mathcal{I}] = 1$, where $q = q_\mu$. $\square$

Proving $\lambda_c^{AB}(r) \leq \lambda_c(2r)$ (2): Coupling Lemma
If $\mathbb{P}_{p_\nu,1,\varepsilon}[G(\mathcal{A},\mathcal{B},t) \in \mathcal{I}] = 1$ then $\exists q < 1$:

$$\mathbb{P}_{p_\lambda,q,\varepsilon}[G(\mathcal{A},\mathcal{B},t) \in \mathcal{I}] = 1.$$

Proof: Consider a Bernoulli random field of 'open' vertices and edges of the directed graph (V,E) with $V = \varepsilon\mathbb{Z}^d$ and $(u,v) \in E$ iff $|u-v| \leq t$.

Each vertex $v \in V$ is open with probability p_λ and each edge (u,v) is open with probability ϕ (chosen below). Deine the subsets of V:

$\mathcal{A}_1 := \{v : v$ is open and all edges out of v are open$\}$
$\mathcal{B}_1 = \varepsilon\mathbb{Z}^d$;

$\mathcal{A}_2 = \{v : v$ is open $\}$
$\mathcal{B}_2 = \{v :$ at least one edge into v is open$\}$.

If $G(\mathcal{A}_1,\mathcal{B}_1,t) \in \mathcal{I}$ then $G(\mathcal{A}_2,\mathcal{B}_2,t) \in \mathcal{I}$.

Can choose ϕ so $\mathbb{P}[v \in \mathcal{A}_1] = p_\nu$. Then by our assumption, $G(\mathcal{A}_1,\mathcal{B}_1,t)$ percolates and hence so does $G(\mathcal{A}_2,\mathcal{B}_2,t)$. $\square$

A finite bipartite geometric graph

Set $d = 2$. Set $\mathcal{P}_\lambda^F = \mathcal{P}_\lambda \cap [0,1]^2$, $\mathcal{Q}_\lambda^F = \mathcal{Q}_\lambda \cap [0,1]^2$. Let $\tau > 0$.

Let $G'(\lambda, \tau, r)$ be the graph on $V = \mathcal{P}_\lambda^F$ with X, X' connected iff they have a common neighbour in $G(\mathcal{P}_\lambda^F, \mathcal{Q}_{\tau\lambda}^F, r)$, i.e.

$$E(G'(\lambda,\tau,r)) = \{\{X,X'\} : \exists Y \in \mathcal{Q}_{\tau\lambda}^F \text{ with } |X-Y| \le r, |X'-Y| \le r\}$$

Let $\rho_\lambda(\tau) = \min\{r : G'(\lambda, \tau, r) \text{ is connected }\}$ (a random variable).

THEOREM 2 (MP 2014). $\lambda\pi(\rho_\lambda(\tau))^2/\log\lambda \xrightarrow{P} \frac{1}{\tau \wedge 4}$ as $\lambda \to \infty$.

and with a suitable coupling this extends to a.s. convergence as λ runs through the integers.

Idea of proof. Isolated vertices determine connectivity.

Partial sketch proof of Theorem 2

Let $a > 0$. Suppose $\lambda \pi r_\lambda^2 / \log \lambda = a$.

Let N_λ be the number of isolated points in $G(\mathcal{P}_\lambda^F, \mathcal{Q}_{\tau\lambda}^F, r_\lambda)$.

Let N'_λ be the number of isolated points in $G(\mathcal{P}_\lambda^F, 2r_\lambda)$. On the torus,

$$\mathbb{E}[N_\lambda] = \lambda \exp(-\tau\lambda(\pi r_\lambda^2)) = \lambda^{1-a\tau}.$$

$$\mathbb{E}[N'_\lambda] = \lambda \exp(-\lambda(\pi(2r_\lambda)^2)) = \lambda^{1-4a}.$$

Both expectations go to zero iff $a > 1/\tau$ and $a > 1/4$, i.e. $a > 1/(\tau \wedge 4)$.

'Soft' random geometric graphs

Let $\phi : \mathbb{R}_+ \to [0,1]$ nonincreasing; $d \geq 2$, $\lambda > 0$. Let $G(\lambda, \phi)$ have vertex set $V_\lambda := \mathcal{P}_\lambda \cap [0,1]^d$.

Each $x, y \in V_\lambda$ are connected by an edge with probability $\phi(|y-x|)$ (generalises geometric and Erdos-Renyi random graphs). Then

$$\mathbb{E}[N_0(G(\lambda,\phi))] = \lambda \int \exp\left(-\lambda \int \phi(|y-x|)dy\right) dx,$$

with all integrals being over $[0,1]^d$.

Let $\mathcal{K} :=$ { connected $G : 2 \leq |V(G)| < \infty$}
Let $\mathcal{M}_1 := \{G$ with $N_0(G) = 0\}$
$N_0(G) :=$ # isolated vertices of G. Clearly

$$\mathcal{K} \subset \mathcal{M}_1.$$

Might hope that for $G = G(\lambda, \phi)$ with λ large

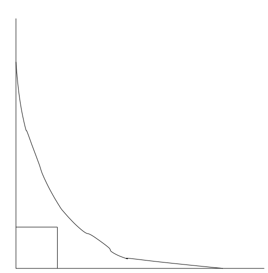
$$\mathbb{P}[G \in \mathcal{K}] \approx \mathbb{P}[G \in \mathcal{M}_1] \approx \exp(-\mathbb{E}[N_0(G)])$$

A class of connection functions

Given decreasing $\phi : \mathbb{R}_+ \to [0, 1]$, and $\eta > 0$, let

$$r_\eta(\phi) = \inf\{t \in \mathbb{R}_+ : \phi(t) \le \eta\phi(0)\},$$
$$r_0(\phi) = \sup\{t \in \mathbb{R}_+ : \phi(t) > 0\}.$$

Let Φ_η be the class of connection functions ϕ with $r_\eta(\phi) \ge \eta r_0(\phi)$.



Given η, Φ_η is a class of connection functions that have uniformly finite range measured in terms of their characteristic length-scale $r_\eta(\phi)$.

Note $\Phi_\eta \subset \Phi_{\eta'}$ for $\eta > \eta'$.

All step functions of the form $\phi(t) = p\mathbf{1}_{[0,r]}(t)$ are in Φ_1.

Limit theorem for soft RGGs. (MP 2016). Suppose $d \geq 2$, $\eta > 0$. Then as $\lambda \to \infty$,

$$\sup_{\phi \in \Phi_\eta} \Big| \mathbb{P}[G(\lambda,\phi) \in \mathcal{M}_1] - \exp\left[-\lambda \int \exp\left(-\lambda \int \phi(|y-x|)dy\right) dx\right] \Big| \to 0$$

(where all integrals are over $[0,1]^d$) and

$$\sup_{\phi \in \Phi_\eta} \Big| \mathbb{P}[G(\lambda,\phi) \in \mathcal{K}] - \exp\left[-\lambda \int \exp\left(-\lambda \int \phi(|y-x|)dy\right) dx\right] \Big| \to 0.$$

Thus

$$\sup_{\phi \in \Phi_\eta} |\mathbb{P}[G(\lambda,\phi) \in \mathcal{K}] - \mathbb{P}[G(\lambda,\phi) \in \mathcal{M}_1]| \to 0.$$

Also a de-Poissonized version of this result holds. (with V_λ replaced by n i.i.d. points in $[0,1]^d$).

Domination number of random geometric graphs

A *dominating set* in a graph $G = (V, E)$ is a set $S \subset V$ such that $\mathrm{dist}(S, V) \leq 1$.

Domination number $\gamma(G) = \min\{|S| : S$ a dominating set$\}$.

Theorem. Suppose $d = 2, \lambda^{-1/2} \ll r_\lambda \ll 1$ as $\lambda \to \infty$. Then

$$\pi r_\lambda^2 \gamma(G(\mathcal{P}_\lambda^F, r_\lambda)) \xrightarrow{P} C = 2\pi\sqrt{3}/9 \approx 1.209$$

as $\lambda \to \infty$. If instead $\lambda r_\lambda^2 \to \mu \in (0, \infty)$,

$$\pi r_\lambda^2 \gamma(G(\mathcal{P}_\lambda^F, r_\lambda)) \xrightarrow{P} H(\mu)$$

for some $H(\mu)$. [Cf. Bonato, Lozier, Mitsche, Peréz-Gimenéz, Prałat 2015]

What about the soft graph $G(\mathcal{P}_\lambda^F, r_\lambda, p)$ for p fixed?