A universality theorem in random matrix theory

Manjunath Krishnapur
Joint work with Brian Rider and Bálint Virág

Indian Institute of Science

6 Oct 2016

The beta-log gas and the main question

Joint density of n particles $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the line:

$$p_{n,\beta}^V(\lambda) := \frac{1}{Z_{n,\beta}^V} e^{-\beta H(\lambda_1,\ldots,\lambda_n)}$$

where $\beta > 0$ and $V(x) = x^{2p}$ (or any strictly convex polynomial)

$$H(\lambda_1, \ldots, \lambda_n) = n \sum_{k=1}^{n} V(\lambda_k) - \sum_{i<j} \log |\lambda_i - \lambda_j|.$$

Question

Let $\lambda_n^* = \max_{k \le n} \lambda_k$. Do there exist $loc_n \in \mathbb{R}$, $scl_n > 0$ such that

$$\frac{\lambda_n^* - loc_n}{scl_n} \xrightarrow{d} \textit{lim-distr.}_{V,\beta}?$$

loc_n? scl_n? lim-distr.$_{\beta,V}$? Is the limit distribution same for all V?

Our answer

Definition

Let W be standard Brownian motion and define the *stochastic Airy operator*

$$H_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} W'(x)$$

on $L^2(\mathbb{R}^+)$. It has (random) eigenvalues $\theta_1 < \theta_2 < \ldots \to \infty$.

Theorem

$n^{2/3}(\lambda_n^* - R_V) \xrightarrow{d} \theta_1$ *where* R_V *is a number defined in terms of* V *alone.*

Why? A bit of broad context: Boltzmann's recipe

Take three ingredients

- Configuration space Ω and a reference measure ν on Ω.
- Energy function/Hamiltonian $H : \Omega \mapsto \mathbb{R}$.
- Inverse temperature $\beta > 0$.

and create the probability mass function or density (w.r.t. ν)

$$p(\mathbf{x}) = \frac{1}{Z_\beta} e^{-\beta H(\mathbf{x})}.$$

This prescription incorporates a good fraction of probability distributions that probabilists study.

Why? A bit of broad context: Examples

- *A particle in a quadratic potential well*: $\Omega = \mathbb{R}$ and $H(x) = x^2/2$. This gives Gaussian distribution

$$p(x) \propto e^{-\beta x^2} \quad \text{w.r.t. Lebesgue measure.}$$

- *Brownian motion*: $\Omega = C[0,1]$ and $H(f) = \frac{1}{2}\int_0^1 (f'(x))^2 dx$.

$$p(f) \propto e^{-\beta \int_0^1 (f'(t))^2 dt} \quad \text{w.r.t. ?????}$$

- *Non-interacting particles in a potential well*: $\Omega = \mathbb{R}^n$ and $H(x_1, \ldots, x_n) = \sum_{i=1}^n V(x_i)$.

$$p(\mathbf{x}) \propto \prod_{k=1}^n e^{-V(x_k)} \quad \text{w.r.t. Lebesgue measure on } \mathbb{R}^n.$$

  Gives independence in the sense of probability. Distribution of the right-most point well-understood.

Coulomb potential

Coulomb's law: A unit charge at location $\mathbf{x}_0$ gives rise to a potential $|\mathbf{x}-\mathbf{x}_0|^{-1}$. In dimension d, it should be modified to

$$G(\mathbf{x},\mathbf{x}_0) = \begin{cases} |\mathbf{x}-\mathbf{x}_0|^{-d+2} & \text{if } d \geq 3, \\ \log|\mathbf{x}-\mathbf{x}_0|^{-1} & \text{if } d = 2, \\ |\mathbf{x}-\mathbf{x}_0| & \text{if } d = 1. \end{cases}$$

Reason: $\Delta_{\mathbf{x}} G(\mathbf{x},\mathbf{x}_0) = c_d \delta_{\mathbf{x}_0}$ (Green's function for Laplacian).

Superposition principle: Multiple particles interact pairwise.

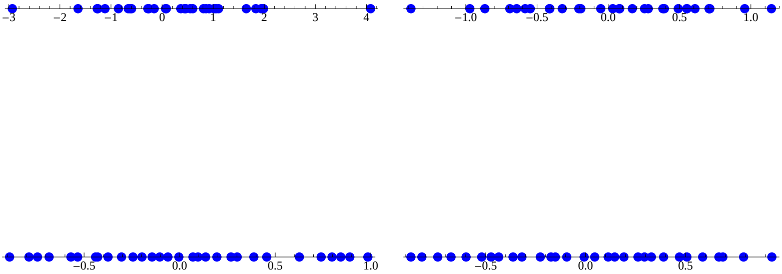
Now let $d = 1$ but use the $2d$-potential. Place n unit charges in potential well $nV(\cdot)$. Then,

$$H(\lambda_1,\ldots,\lambda_n) = n\sum_{k=1}^{n} V(\lambda_k) - \sum_{i<j} \log|\lambda_i - \lambda_j|.$$

The corresponding density $p_{n,\beta}^{V}(\cdot)$ is what we want to study.

Beta gases with quadratic potential

$n = 25$ particles, $V(x) = x^2$, four values of β. Higher the β, higher the repulsion between points. The case $\beta = 2$ is particularly special.



Bulk shape (prelude to finding location and scale of λ_n^*)

Write the density as

$$
\begin{aligned}
-\log p(\lambda) &= \beta n \sum_{k=1}^{n} V(\lambda_k) - \sum_{i<j} \log|\lambda_i - \lambda_j| - \log Z_{\beta,n}^{V} \\
&= \beta n^2 \left\{ \frac{1}{n} \sum_{k=1}^{n} V(\lambda_k) - \frac{1}{2n^2} \sum_{i \neq j} \log|\lambda_i - \lambda_j| \right\} - \cdots
\end{aligned}
$$

The most likely λ is the one that minimizes this quantity. How does it look? What about typical λ?

Perhaps more familiar: If n charges are confined to an interval, where do they settle? *Answer [Stieltjes]*: At the zeros of the Legendre polynomial!. If allowed to roam anywhere on the whole line but with a potential nx^2 applied, the most likely configuration is the set of zeros of the (properly scaled) Hermite polynomial.

Bulk shape (prelude to finding location and scale of λ_n^*)

Theorem (Known)

Assume $V(x) \gg \log|x|$. There is a unique probability measure μ_V that minimizes

$$\mathcal{L}[\mu] := \int V(x)d\mu(x) - \frac{1}{2}\iint \log|x-y|d\mu(x)d\mu(y).$$

Further, $\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k} \xrightarrow{P} \mu_V$, as $n \to \infty$. (More: LDP)

- Example 1: $V(x) = x^2$. Then $d\mu_V(x) = \frac{1}{\pi}\sqrt{4-x^2}dx$ (semicircle law).
- Example 2: $V(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \infty & \text{if } |x| > 1. \end{cases}$ μ_V is arcsine law.

Finding the location and scale of λ_n^*

Fact: If V is uniformly convex, then μ_V is supported on a single interval $[L_V, R_V]$. It has density ρ_V that is positive on this interval, smooth inside and vanishes like $\sqrt{R_V - x}$ at the right edge.

Reasonable expectation: λ_n^* is located close to R_V in a window of length $n^{-2/3}$. In other words,

$$n^{2/3}(\lambda_n^* - R_V)$$

may have a non-trivial limiting distribution.

Non-convex potentials: Many strange things are possible. Can find polynomial V such that ρ_V vanishes like $(R_V - x)^{3/2}$ at the edge.Then, $n^{2/5}(\lambda_n^* - R_V)$ is the right thing to consider. For $\beta = 2$ these have been proved to be valid.

Summary so far

Let $\beta > 0$, V a uniformly convex polynomial of even degree. For $n \geq 1$ define the probability density

$$p_{n,\beta}^{V}(\lambda) \propto \exp\left\{-\beta\left[n\sum_{k=1}^{n} V(\lambda_k) - \sum_{j<k} \log|\lambda_j - \lambda_k|\right]\right\}$$

and let $\lambda_n^* = \max\{\lambda_1, \ldots, \lambda_n\}$. There is a unique minimizer μ_V of

$$\mathcal{L}[\mu] := \int V(x)d\mu(x) - \frac{1}{2}\iint \log|x-y|d\mu(x)d\mu(y)$$

that is supported on a single interval $[L_V, R_V]$. We expect that

$$n^{2/3}(\lambda_n^* - R_V)$$

has a limit distribution on the line (and in fact the same must be true for the second eigenvalue, third, ...)

The universality hypothesis: This limit distribution does not depend on V.

Methods of approach

- $\beta = 2, V(x) = x^2$. First done by Tracy and Widom. The now famous Tracy-Widom distribution was discovered. Method: Fine analysis of Hermite polynomials (too vague).
- $\beta = 2$, general V (even beyond convex or polynomial). Many works of integrable systems people, mainly Percy Deift and many collaborators. Method: Riemann-Hilbert techniques.
- $\beta > 0$, $V(x) = x^2$. Edelman-Sutton and Dumitriu (heuristics). Ramirez-Rider-Virág. Method: Tridiagonal random matrices and stochastic operator limits.
- $\beta > 0$, general V
  (a) Bourgade-Erdös-Yau (heat flow),
  (b) Bekerman, Figalli, Guionnet (mass transportation),
  (c) K., Rider, Virág (tridiagonal matrices, operator limits)

Skip: Parallel story of bulk universality

Tridiagonal matrices - a quick introduction

Let $(a,b) = (a_1, \ldots, a_n, b_1, \ldots, b_{n-1})$ where $a_k \in \mathbb{R}$ and $b_k > 0$. Define

$$T = T_n(a,b) = \begin{bmatrix} a_1 & b_1 & 0 & 0 & \ldots & 0 \\ b_1 & a_2 & b_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & \ldots & 0 & b_{n-1} & a_n \end{bmatrix}$$

Let $\sum_{k=1}^{n} q_k^2 \delta_{\lambda_k}$ denote the spectral measure of T_n at the first co-ordinate vector. This means that for all $p \geq 0$,

$$(T^p)_{1,1} = \sum_{k=1}^{n} \lambda_k^p q_k^2.$$

The following association is a bijection (almost).

$$\mathbb{R}^n \times \mathbb{R}_+^{n-1} \ni (a,b) \longleftrightarrow (\lambda_1, \ldots, \lambda_n, q_1^2, \ldots, q_n^2) \in \mathbb{R}^n \times \text{Symplex}_n$$

The tridiagonal matrix model

Lemma (Trotter, Dumitriu-Edelman, KRV)

If (a, b) has the density

$$\exp\left\{-n\beta\left[tr(V(T(a,b))) - \sum_{k=1}^{n-1}(1-\frac{k}{n}-\frac{1}{n\beta})\log b_k\right]\right\},$$

then λ and q are independent,

$$(q_1^2, \ldots, q_n^2) \sim Dirichlet(\beta/2, \ldots, \beta/2),$$
$$(\lambda_1, \ldots, \lambda_n) \sim p_{n,\beta}^V \quad (\textit{the log-gas}).$$

Important special case: When $V(x) = x^2$, the density of (a, b) is better rephrased as:

$$\sqrt{\beta n}\, a_k \sim N(0,2), \quad \beta n\, b_k^2 \sim \chi^2_{\beta(n-k)}$$

and all the a_is and b_js are independent.

How does the tridiagonal matrix help?

If $T = T_n(a, b)$, then it acts on $\mathbf{x} \in \mathbb{R}^n$ as a difference operator

$$(T\mathbf{x})_k = a_k x_k + b_{k-1} x_{k-1} + b_k x_{k+1}.$$

We may hope that in an appropriate scaling, it will converge to a differential operator.

Example: If $a_k = -2$, $b_k = 1$ for all k, then

$$(T\mathbf{x})_k = (x_{k+1} - x_k) - (x_k + x_{k-1}).$$

If $x_k = f(k/n)$ for a nice function f, then $n^2 (T\mathbf{x})_{[nt]} \approx f''(t)$.

General slogan in probability Prove distributional limit for the largest possible object. All its features will also have distributional limits. [Best known example: Donsker's invariance principle]. Here, we wanted only largest eigenvalue. But we prove convergence at the level of operators.

Description of the limit operators

Definition

Let W be standard Brownian motion and define the *stochastic Airy operator*

$$H_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} W'(x)$$

on $L^2(\mathbb{R}^+)$. It has (random) eigenvalues $\theta_1 < \theta_2 < \ldots < \theta_n \to \infty$ with eigenfunctions $f_1, f_2, \ldots$ defined by variational formulas on the quadratic form

$$Q[f,f] = \int_0^\infty [f'(x)^2 + x^2 f^2(x)]dx + 2\sqrt{\beta}\int_0^\infty f^2(x)dW(x)$$

for $f \in L^2(\mathbb{R}_+)$ satisfying $f' \in L^2(\mathbb{R}_+)$, $xf(x)^2 \in L^2(\mathbb{R}_+)$ and $f(0) = 0$.

Operator convergence

Theorem

One can couple (a, b) have the joint density given earlier and H_β so that (for suitable constants γ, γ')

$$\gamma n^{2/3}(R_V - T_n) \xrightarrow{a.s.} H_\beta \qquad (\textit{in norm-resolvent sense})$$

where R_V is the right edge of the support of μ_V. Consequently

- $\lambda_n^* \xrightarrow{d} \theta_1$ *(and so on for the second, third,... eigenvalues).*
- *If $\mathbf{v}_n$ is the top eigenvector of T_n, then $t \mapsto (\gamma' n)^{1/6}\mathbf{v}_n([\gamma' t n^{-1/3}])$ converges in distribution to f_1.*