On the boundary of the support of super-Brownian motion

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Stoshastic heat equation

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t),$$

where $\dot{W}$ is the Gaussian space-time white noise with

$$E\left[\dot{W}(x,t)\dot{W}(y,s)\right] = \delta(t-s)\delta(x-y).$$

$$\begin{aligned} X(t,x) &= \int p_t(x-y)X(0,y)dy \\ &\quad + \int_0^t \int p_{t-s}(x-y)\sigma(X(s,y))\, W(dy,ds). \end{aligned}$$

Uniqueness

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

- **Pathwise uniqueness (PU):**
  X^1, X^2 — two solutions, $X^1(0,\cdot) = X^2(0,\cdot)$
  $\Longrightarrow X^1(t,\cdot) = X^2(t,\cdot)\,, \forall t > 0.$

- **Uniqueness in law (weak):**
  X^1, X^2 — two solutions (even on different spaces),
  $X^1(0,\cdot) = X^2(0,\cdot) \Longrightarrow \{X^1(t,\cdot)\}_{t\geq 0} \stackrel{law}{=} \{X^2(t,\cdot)\}_{t\geq 0}\,.$

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + \sigma(X(t,x))\dot{W}(x,t).$$

If $\dot{W}$ is a space-time white noise, then function-valued solution exists if $d = 1$.
Uniqueness?
σ — Lipschitz $\Longrightarrow$ PU follows easily.
σ - non-Lipschitz ?

Super-Brownian motion

Branching Brownian motions in $\mathbf{R}^d$.
X^n:
$\sim n$ particles in $\mathbf{R}^d$ at time 0.
$\frac{1}{n}, \frac{2}{n}, \ldots$ — times of death or split,
$p_0 = p_2 = \frac{1}{2}$ — probabilities of death or split.
Critical branching: mean number of offspring $= 1$.
New particles move as independent Brownian motions.

$$X_t^n(A) = \frac{\#\text{ particles in } A \text{ at time } t}{n}, \quad A \subset \mathbf{R}^d.$$
$$X^n \Rightarrow X,$$

X is a super-Brownian motion — measure-valued process.

Characterization

Laplace transform:

$$E\left[e^{-\langle X_t,\phi\rangle}\right] = E\left[e^{-\langle X_0,u_t\rangle}\right], \quad \phi \geq 0.$$

where

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t - \frac{1}{2}u_t^2, \quad u_0 = \phi.$$

X is continuous (in time) measure-valued process.

Regularity properties?

Properties of SBM

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  $\dot{W}$ — Gaussian space-time white noise.
- From now on

$$d = 1.$$

More properties

- $$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

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  Hölder $1/2-$ in space, Hölder $1/4-$ in time.

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- Define:

$$\begin{aligned} BZ_t &\equiv \partial(\{x : X(t,x) = 0\}) \\ &= \{x : X(t,x) = 0, \forall \delta > 0\, X_t((x-\delta, x+\delta)) > 0\}. \end{aligned}$$

  — the boundary of the zero set of X_t (or boundary of the support of X_t).

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- Question:

  Properties of BZ_t?

Hausdorff dimension

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

In particular we are interested in Hausdorff dimension of BZ_t:

$$\dim(BZ_t) = ?$$

Motivation: Pathwise Uniqueness for SBM?

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

Weak uniqueness holds (by duality method)

Pathwise uniqueness (PU)?
$\sqrt{X}$ — non-Lipschitz.

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Is there a chance to get **PU**?

Pathwise uniqueness for SDEs

$$dX_t = \sigma(X_t)dB_t$$

B_t is a one-dimensional Brownian motion.

Theorem (Yamada, Watanabe (71))

If σ is Hölder continuous with exponent $1/2$, then PU holds.

Remark

There are counter examples for σ which is Hölder continuous with exponent less than $1/2$.

Back to SPDEs

Theorem (Perkins, M., 11)

Let $\sigma(x)$ be Hölder continuous with exponent γ.
For any $\gamma > 3/4$, **PU** *holds for*

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

where $\dot{W}$ is space-time white noise.

Ingredients of the proof

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sigma(X)\dot{W},$$

X^1, X^2 — two solutions, $\tilde{X} = X^1 - X^2$.

$$\frac{\partial \tilde{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \tilde{X}_t(x) + (\sigma(X_t^1(x)) - \sigma(X_t^2(x)))\dot{W}(t,x).$$

Clearly

$$|\sigma(X_t^1(x)) - \sigma(X_t^2(x))| \leq C|\tilde{X}_t(x)|^\gamma,$$

and thus one can show that the that it is enough to consider uniqueness of

$$\frac{\partial \bar{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \bar{X}_t(x) + |\bar{X}_t(x)|^\gamma \dot{W}(t,x).$$

Regularity and uniqueness of $\bar{X}$

$$\frac{\partial \bar{X}_t(x)}{\partial t} = \frac{1}{2}\Delta \bar{X}_t(x) + |\bar{X}_t(x)|^\gamma \dot{W}(t,x).$$

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- $x \mapsto \bar{X}_t(x)$ is Hölder $1/2 - \epsilon$.
- For $x \in BZ_t$, roughly we have

  $x \mapsto \bar{X}_t(x)$ is Hölder with any exponent

$$\xi < \frac{1}{2(1-\gamma)}.$$

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- Put it together: PU holds if

$$\gamma > 3/4.$$

Non-uniqueness

- Is 3/4 sharp? Counter example: for $\gamma < 3/4$ try to construct non-triviual solution to

$$\begin{cases} \frac{\partial}{\partial t} X(t,x) &= \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma} \dot{W}(x,t), \\ X(0,\cdot) &= 0. \end{cases} \tag{1}$$

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- Burdzy, Mueller, Perkins(2010); M., Mueller, Perkins(2012): If $0 < \gamma < 3/4$ there is solution $X(t,x)$ to (1) such that with positive probability, $X(t,x)$ is not identically zero.

All this was about any solution to the SPDE.
What happens if we restrict consideration to the class of non-negative solutions?

Non-uniqueness for non-negative solutions

- Burdzy, Mueller, Perkins(2010): If $0 < \gamma < 1/2$, $\psi \geq 0$, non-trivial, then **PU** fails for **non-negative** solutions to

$$\frac{\partial}{\partial t}X(t,x) = \frac{1}{2}\Delta X(t,x) + |X(t,x)|^{\gamma}\dot{W}(x,t) + \psi, \quad (2)$$

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This is super-Brownian motion with immigration ψ for which **weak uniqueness** holds!
Presence of ψ is very important: whenever $\psi > 0$, boundary of the zero set of any solution has positive Lebesgue measure Heuristically, it is "easier" for two solutions to separate if the boundary of the "zero set" is "large".

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  Heuristically, it is "easier" for two solutions to separate if the boundary of the "zero set" is "large".
- It is conjectured that Chen's argument could be extended for $\gamma < 3/4$ case.

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As we mentioned the presence of ψ is very important: whenever $\psi > 0$, BZ_t of any solution has positive Lebesgue measure.

However if the set BZ_t is "small" then one may expect that PU holds also for some $\gamma < 3/4$.

This motivated our interest in the Hausdorff dimension of the boundary of the zero set of $X(t,\cdot)$ that solves (4).
At this point we can do it only in $\gamma = 1/2$ case: SBM without immigration.

Hausdorff dimension of BZ_t fir SBM without immigration

$$\frac{\partial X}{\partial t} = \frac{1}{2}\Delta X + \sqrt{X}\dot{W}.$$

Main question: Hausdorff dimension of BZ_t

$$\dim(BZ_t) = ?$$

What one might expect?

- We have mentioned:

$$x \mapsto X_t(x) \text{ is Hölder } 1/2 - \epsilon.$$

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- Thus, for a long time we believed in conjecture

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- The last conjecture is false.

Theorem 1

There exists $\eta \in (0,1)$, *such that,* $\forall t > 0, x \in R$

$$P(0 < X(t,x) \le \epsilon) \sim \epsilon^{\eta}, \quad \text{as } \epsilon \downarrow 0.$$

Theorem 2

For all $t > 0$

$$\dim(BZ_t) \le 1 - \eta, \qquad P - a.s.$$

and with positive probability,

$$\dim(BZ_t) \ge 1 - \eta, \qquad \text{on } \{X_t(R) > 0\},$$

where η *is from Theorem 1.*

Throughout the proofs we will get the vaue of η.

Proofs

By a Tauberian theorem

$$P(0 < X(t,x) \leq \epsilon) \sim \epsilon^{\eta}, \quad \text{as } \epsilon \downarrow 0,$$

iff

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda^{-\eta}, \quad \text{as } \lambda \uparrow \infty,$$

That is we need to study the assymptotic behavior of

$$\begin{aligned} & E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \\ = \quad & E(e^{-\lambda X(t,x)}) - P(X(t,x) = 0), \quad \text{as } \lambda \uparrow \infty, \end{aligned}$$

Main question: assymptotic behavior of

$$E(e^{-\lambda X(t,x)}) - P(X(t,x) = 0), \text{ as } \lambda \uparrow \infty.$$

Let V^λ be solution of log-Laplace equation with initial condition $V_0 = \lambda\delta_0$. That is

$$\frac{\partial V_t^\lambda}{\partial t} = \frac{1}{2}\Delta V_t^\lambda - \frac{1}{2}(V_t^\lambda)^2, \quad V_0^\lambda = \lambda\delta_0.$$

For simplicity, let $X_0 = \delta_0$. Then it is easy to check that

$$E_{\delta_0}(e^{-\lambda X(t,x)}) = e^{-V^\lambda(t,x)},$$

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Thus

$$\begin{aligned} E_{\delta_0}(e^{-\lambda X(t,x)}) - P_{\delta_0}(X(t,x) = 0) &= e^{-V^\lambda(t,x)} - e^{-V^\infty(t,x)} \\ &\sim V^\infty(t,x) - V^\lambda(t,x). \end{aligned}$$

Very Single Solution (VSS)

$$V^{\infty}(t,x) = \lim_{\lambda \to \infty} V^{\lambda}(t,x), \forall (t,x) \in R_+ \times R \setminus \{(0,0)\}.$$

V^{∞} is called *very singular solution (VSS)* to log-Laplace equation (Brezis, Peletier, Terman(86)).

One can easily check (BPT(86)) that $V = V^{\infty}$ is a $C^{1,2}$ (on $R_+ \times R \setminus \{(0,0)\}$) solution of

$$\begin{aligned} &(i) \quad \frac{\partial V}{\partial t} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2}V^2 \\ &(ii) \quad V(0,x) = 0 \text{ for all } x \neq 0; \ \lim_{t \to 0} \int_R V(t,x)\, dx = \infty. \end{aligned} \tag{5}$$

The analysis of $P(0 < X(t,x) \le \epsilon) \sim \epsilon^\eta, \ \text{as } \epsilon \downarrow 0$, boils down to to the analysis of behaviour of

$$V^\infty(t,x) - V^\lambda(t,x), \ \text{as } \lambda \uparrow \infty.$$

Another simple reduction shows that in fact

$$V^\infty(t,x) - V^\lambda(t,x) \quad \sim \quad \lambda \frac{\partial V^\lambda(t,x)}{\partial \lambda} =: \lambda U^\lambda(t,x),$$

where U^λ solves the following equation

$$\frac{\partial U_t^\lambda}{\partial t} \quad = \quad \frac{1}{2}\Delta U_t^\lambda - V_t^\lambda U_t^\lambda, \quad U_0^\lambda = \delta_0.$$

Therefore by Feynman-Kac and reversing the time we get

$$U^\lambda(t,x) \quad \approx \quad E_0(e^{-\int_0^t V^\lambda(s,W_s)ds})$$

Analysis of behavior of U^λ

$$U^\lambda(t,x) \quad \sim \quad E_0(e^{-\int_0^t V^\lambda(s,W_s)ds})$$

Afer scaling and transformations

$$B_t \quad = \quad \lambda W_{\lambda^{-2}t}, \quad Y(t) = B(e^t - 1)e^{-t/2}$$

we get

$$\begin{aligned} U^\lambda(t,x) \quad &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{e^{s/2}}(1,Y_s)ds}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{\infty}(1,Y_s)ds}) \end{aligned}$$

where Y is an Ornstein-Uhlenbeck process with generator

$$Lh(x) \quad = \quad \frac{1}{2}h''(x) - \frac{1}{2}xh'(x).$$

Analysis of behavior of U^λ

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Let

$$F(x) \equiv V^\infty(1, x).$$

$$U^\lambda(t, x) \sim E_0(e^{-\int_0^{\log(\lambda^2 t)} F(Y_s)ds}).$$

Then

$$\begin{aligned} U^\lambda(t, x) &\sim e^{-\nu_0(\log(\lambda^2 t))} \\ &= \lambda^{-2\nu_0} t^{-\nu_0}, \end{aligned}$$

where ν_0 is the smallest eigenvalue of

$$-L^F h \equiv -(Lh - Fh).$$

One can show: $1/2 < \nu_0 < 1$.

Finishing the proof of Theorem 1

$$U^{\lambda}(t,x) \quad \sim \quad \lambda^{-2\nu_0} t^{-\nu_0}, \ \ \text{as } \lambda \uparrow \infty,$$

Recall that

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda U^{\lambda}(t,x), \ \ \text{as } \lambda \uparrow \infty.$$

Thus

$$E(e^{-\lambda X(t,x)} 1(X(t,x) > 0)) \sim \lambda^{1-2\nu_0} t^{-\nu_0}, \quad \text{as } \lambda \uparrow \infty,$$

and by the Tauberian theorem

$$P(0 < X(t,x) \le \epsilon) \sim \epsilon^{\eta}, \ \ \text{as } \epsilon \downarrow 0,$$

with

$$\eta = 2\nu_0 - 1.$$

$\dim(BZ_t)$

By Theorem 1

$$P(0 < X(t,x) \leq \epsilon) \sim \epsilon^{\eta}, \quad \text{as } \epsilon \downarrow 0,$$

with

$$\eta = 2\nu_0 - 1.$$

Theorem 2 is a corollary of Theorem 1, its proofs and known regularity of X on BZ_t and thus

$$\begin{aligned} \dim(BZ_t) &\leq 1 - \eta \\ &= 2 - 2\nu_0, \quad \text{a.s.} \end{aligned}$$

and

$$\dim(BZ_t) = 2 - 2\nu_0, \quad \text{on } \{X_t(R) > 0\}$$

with positive probability. Note

$$\mathbf{0} < \mathbf{2} - \mathbf{2}\nu_{\mathbf{0}} < \mathbf{1}.$$

Open Problems

- Proving sharp lower bound: P-a.s.

$$\dim(BZ_t) \;=\; 2-2\nu_0, \;\; \text{on } \{X_t(R)>0\}$$

- $$dim(BZ_t)?$$

  for $\gamma \neq 1/2$.
  Conjecture $\dim(BZ_t) \downarrow$ as $\gamma \uparrow$.
- Uniqueness/non-uniqueness of non-negative solutions to

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial t}X(t,x) & = & \frac{1}{2}\Delta X(t,x) + X(t,x)^{\gamma}\dot{W}(x,t), \\ X(0,\cdot) & \geq & 0. \end{array} \right.$$

  for some $\gamma < 3/4$.

Thank You

Very Singular Solution (VSS)

If we define

$$F(x) = V^{\infty}(1, x),$$

Then it is known that

$$V^{\infty}(t, x) = t^{-1} F\left(\frac{x}{\sqrt{t}}\right).,$$

and F solves ode

$$\begin{cases} \frac{1}{2}F''(x) - \frac{1}{2}F^2(x) + \frac{1}{2}F'(x) + F(x) = 0 \\ F > 0 \\ F'(0) = 0, F(x) \sim c_0 y e^{-y^2/2}, asy \to \infty \end{cases} \tag{6}$$

Analysis of behavior of U^λ

$$U^\lambda(t,x) \quad \sim \quad E_0(e^{-\int_0^t V^\lambda(s,W_s)ds})$$

Scaling of V^λ:

$$V^\lambda(t,x) \quad = \quad \lambda^2 V^1(\lambda^2 t, \lambda x) \tag{7}$$

Define

$$B_t \quad = \quad \lambda W_{\lambda^{-2}t}, \quad Y(t) = B(e^t - 1)e^{-t/2}$$

Then

$$\begin{aligned} U^\lambda(t,x) \quad &\sim \quad E_0(e^{-\int_0^t \lambda^2 V^1(\lambda^2 s, \lambda W_s)ds}) \\ &= \quad E_0(e^{-\int_0^{\lambda^2 t} V^1(u,B_u)du}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{e^{s/2}}(1,Y_s)ds}) \\ &\sim \quad E_0(e^{-\int_0^{\log(\lambda^2 t)} V^{\infty}(1,Y_s)ds}) \end{aligned}$$