Continuum random tree as scaling limit for drainage networks

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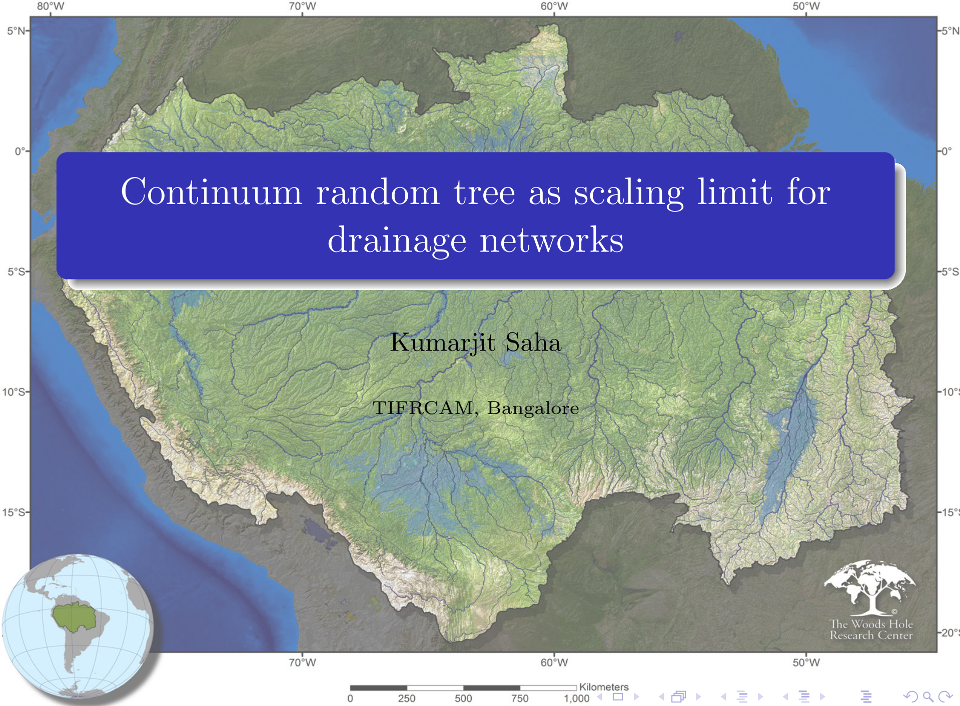
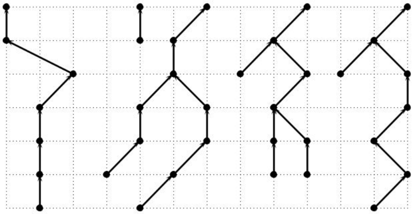


Plan of the talk

- The model and the problem statement
- Motivation behind this problem
- Brief introduction of continuum random tree
- Description of the limiting continuum random tree in our case
- Idea of the proof.
- Universality of this result.

The Howard's model



Each vertex is open with probability p where $p \in (0,1)$.

An open vertex $\mathbf{x} = (\mathbf{x}(1), \mathbf{x}(2))$ connects to the nearest open vertex $\mathbf{y}$ with $\mathbf{y}(2) = \mathbf{x}(2) + 1$.

There is no cycle or loop.

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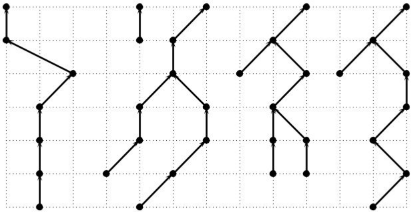

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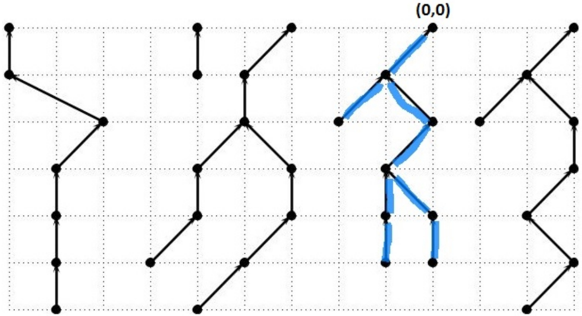
The rooted tree T



T represents the rooted finite tree formed by the collection of all the open vertices whose flows are passing through the origin.

The rooted tree T

Is T finite almost surely?

Theorem (Gangopadhyay, Roy, Sarkar (2004))

With probability 1, there is no bi-infinite path in this random graph.

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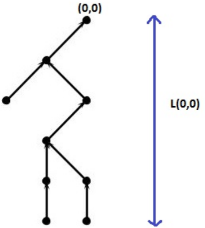


Figure: $T|\{L \geq n\}$ represents a rooted finite tree conditioned to be large.

Does there exist a scaling limit for $T_n := 1/n(T)|\{L \geq n\}$?

Metric space associated with a weighted tree

- A rooted tree has a distinguished vertex called root.
- A weighted tree has strictly positive weights (distances) along each of its edges.
- The distance $d(u, v)$ between any two vertices u, v of the tree is defined as the sum of the weights along the edges on the path (which is unique because of tree structure) from u to v.
- For a rooted weighted tree T and for $c > 0$, cT denote the weighted tree where each edge weight is multiplied by c.

Rooted weighted tree for Howard's model

T gives a rooted weighted tree with root at origin and weight 1 along each edge.

Viewed as a sequence of metric spaces does there exist a limit for $T_n := 1/n(T)|\{L \geq n\}$?

Gromov Hausdorff topology

For a metric space (E, δ), $\delta_{\text{Haus}}(K, K')$ denote the Hausdorff distance between compact subsets K, K' of E :

$$\delta_{\text{Haus}}(K, K') := \inf\{\epsilon > 0 : K \subset U_\epsilon(K') \text{ and } K' \subset U_\epsilon(K)\}. \qquad (1)$$

For T and T' are two rooted weighted trees with roots ρ, ρ', the Gromov Hausdorff distance $d_{GH}(T, T')$ is given by

$$d_{GH}(T, T') := \inf\{\delta_{Haus}(\phi(T), \phi'(T')) \vee \delta(\phi(\rho), \phi'(\rho))\} \qquad (2)$$

where the infimum is taken over all choices of metric spaces (E, δ) and all isometric embeddings $\phi : T \to E$ and $\phi' : T' \to E$.

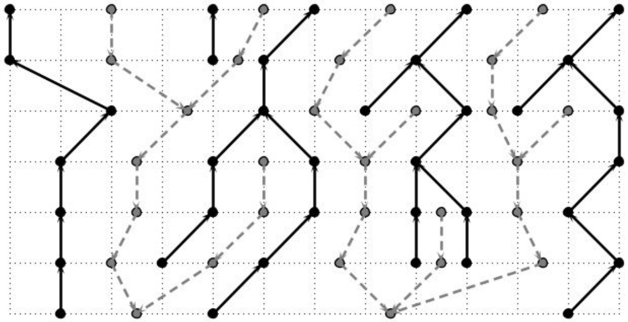
CRT as a scaling limit

Theorem (S. (2016))

As $n \to \infty$, we have

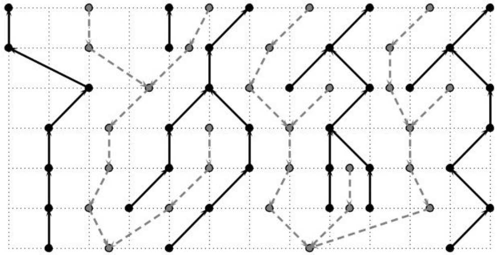
$$1/n(T, \widehat{T})|\{L \geq n\} \Rightarrow (\mathcal{T}_{\mathcal{W}}, \widehat{\mathcal{T}}_{\mathcal{W}})$$

Joint convergence for the dual tree



Since all the clusters are finite, any two dual paths must coalesce in finite time a.s.

Joint convergence for the dual tree



Theorem (S.)

As $n \to \infty$, we have

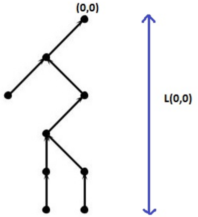
$$1/n(T, \widehat{T}) | \{L \geq n\} \Rightarrow (\mathcal{T}_{\mathcal{W}}, \widehat{\mathcal{T}}_{\mathcal{W}})$$

Motivation: scaling limit of (large) random trees

Scaling limits of large discrete random trees are of independent interest. e.g.

- scaling limit of uniform spanning tree on n vertices
- scaling limit of critical Galton Watson tree with finite variance conditioned to have total population size exactly equal to n
- scaling limit of minimal spanning tree for K_n, complete graph on n vertices

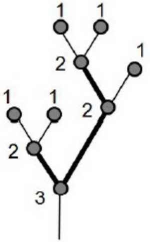
Motivation: self similarity for river networks



T represents a stochastic model of river network.

River networks are empirically observed to be self-similar.

Motivation: Tokunaga's law



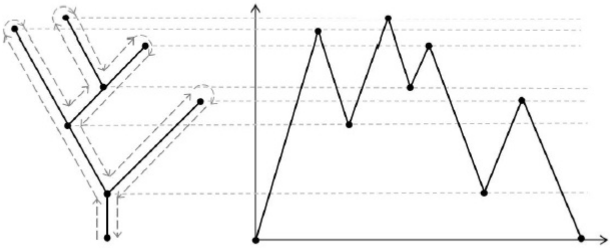
(a) Horton-Strahler orders

According to Tokunaga's law, the average number of j order substreams coming into i order stream $(j < i)$ denoted by $T_{i,j}$ depends only on $i - j$.

$$T_{i,j} \propto f(i - j)$$

Continuum random tree: brief introduction

Loosely speaking continuum tree means a "tree"-like metric space with infinitely many leaves.



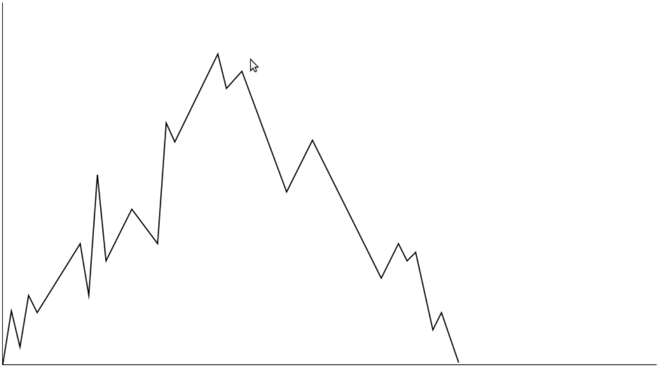
(a) Tree T (b) Harris path H_T

Trees from excursions

Let $h : [0,1] \to \mathbb{R}^+$ be an excursion, that is a continuous function such that $h(0) = h(1) = 0$ and $h(x) > 0$ for $x \in (0,1)$.

Trees from excursions

Now put glue on the underside of the excursion and push the two sides together...



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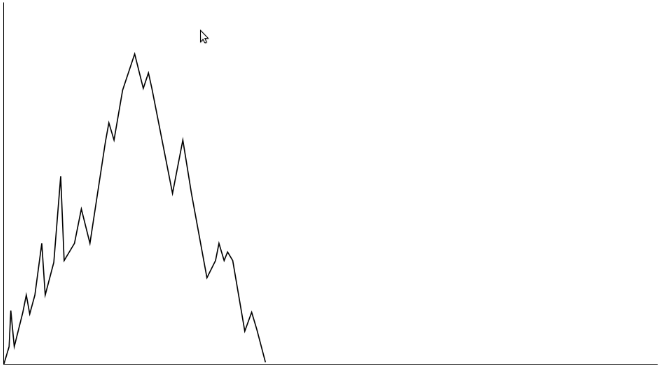

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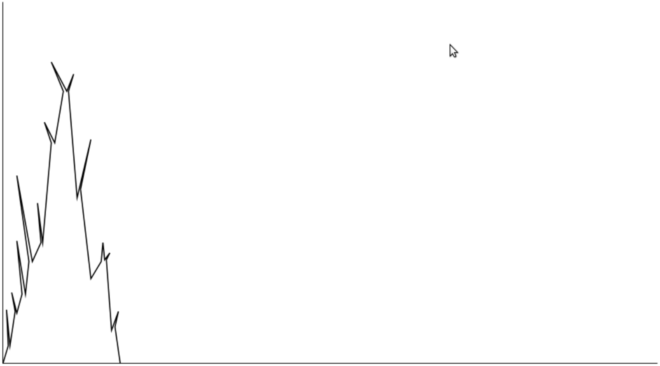

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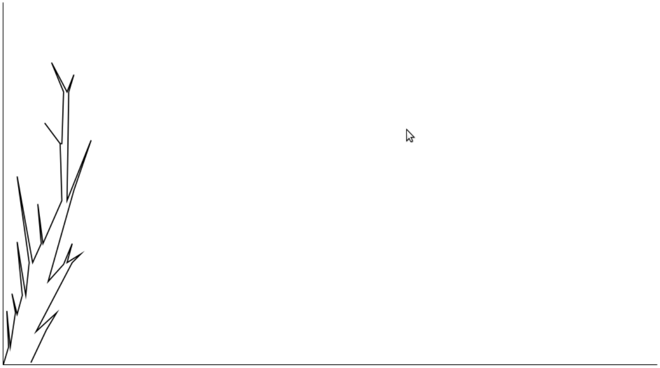
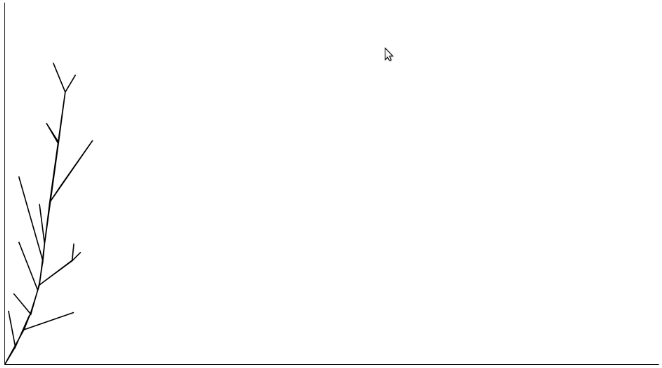
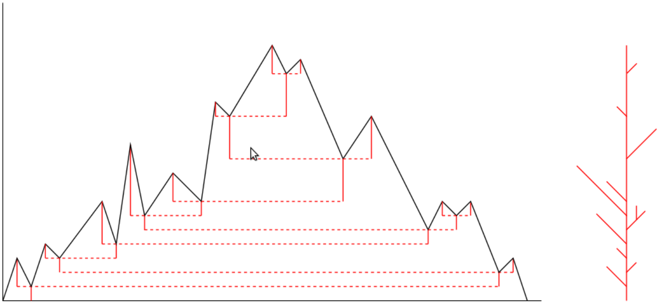


Trees from excursions

Now put glue on the underside of the excursion and push the two sides together to get a tree.



Trees from excursions

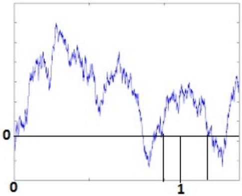


Brownian Excursion W_0^+

Let $\{W(s) : s \geq 0\}$ be a standard Brownian motion with $W(0) = 0$.

Let $\tau_1 := \sup\{s \leq 1 : W(s) = 0\}$ and $\tau_2 := \inf\{s \geq 1 : W(s) = 0\}$.

Note that $\tau_1 < 1$ and $\tau_2 > 1$ almost surely.



The standard Brownian excursion, $W_0^+(s)$, are given by

$$W_0^+(s) := \frac{|W(\tau_1 + s(\tau_2 - \tau_1))|}{\sqrt{\tau_2 - \tau_1}}, \quad s \in [0, 1].$$

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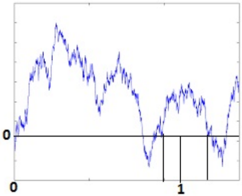

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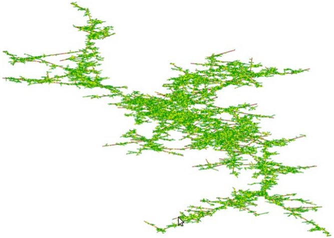
Brownian CRT: the most celebrated CRT

Brownian CRT introduced by Aldous (91), is the continuum random tree whose Harris path is given by a Brownian excursion.



[Picture by Grégory Miermont]

Aldous (93) showed that a properly scaled ($1/\sqrt{n}$ scaling) critical Galton Watson tree with finite variance conditioned to have total population of size n converges to the Brownian CRT.

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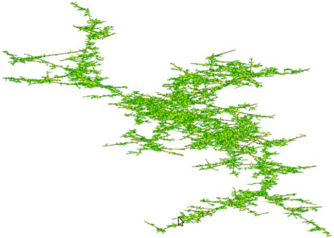

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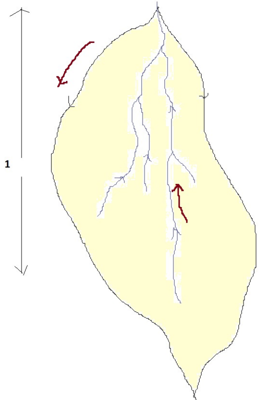
Aldous (93) showed that a properly scaled ($1/\sqrt{n}$ scaling) critical Galton Watson tree with finite variance conditioned to have total population of size n converges to the Brownian CRT.

Description of the limiting CRT

- Consider two independent standard Brownian motions starting at the origin. There will be infinitely many excursions obtained from their intersections.
- Consider the first excursion whose length (time) is more than 1.
- Let Δ denote the random region enclosed by this excursion.

Description of the limiting CRT

Consider collection of coalescing Brownian motions starting from all rational (x, t) in Δ, progressing in the reverse direction of time and following Skorohod reflection once they hit the boundary of Δ.



Description of the limiting CRT

- Soucaliuc, F., Toth, B. and Werner, W.(2000) showed that such a collection is almost surely well defined and its distribution does not depend on the ordering of $\mathbb{Q}^2$.
- This gives a tree like **metric space** with root at the origin.
- $\mathcal{T}_{\mathcal{W}}$ is the completion of this tree like metric space.
- This is different from taking closure in the usual path space topology.

Aldous's conjecture (93) : Scheidegger model

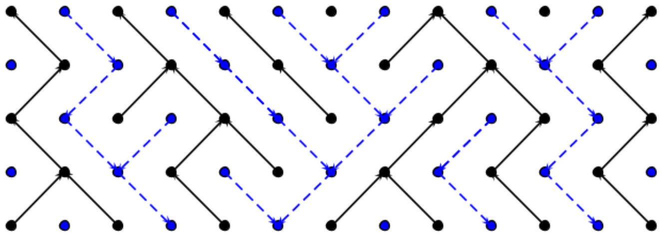


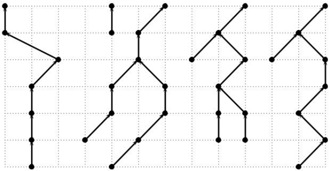
Figure: Scheidegger model gives a system of coalescing nearest neighbour random walks starting from every points on the even lattice. Arratia(79) observed that it has a natural dual which has the same distribution as coalescing simple symmetric random walks starting from every points on the odd lattice.

Aldous conjectured a similar continuum random tree with slight modification. He considered Δ region enclosed by two independent Brownian motions starting at the origin conditioned to meet for the first time exactly at time 1 because he was interested about the scaling limit of discrete tree conditioned to have time-length exactly n.

- We prove this for non degenerate conditional setting $L \geq n$and obtain an universality type result.
- Aldous did not observe Skorohod reflection.
- Some more work is required to work with the degenerate conditioning $\{L = n\}$.

Idea of the proof: Collection of paths obtained from G

Let Π be the collection of all continuous paths in $\mathbb{R}^2$ with all possible starting times such that $\pi \in \Pi$ with starting time $\sigma_\pi \in \mathbb{R}$ is a continuous mapping $\pi : [\sigma_\pi, \infty) \to \mathbb{R}$.



Let $\mathcal{X} := \{\pi^{(x,t)} : (x,t) \in \mathbb{Z}^2_{\text{even}}\}$ denote the collection of all paths obtained from G.
For $n \geq 1$, the scaled path $\pi_n : [\sigma_\pi/n, \infty) \to (-\infty, \infty)$ is given by $\pi_n(t) := \pi(nt)/\sqrt{n}$. Let $\mathcal{X}_n := \{\pi_n^{(x,t)} : (x,t) \in \mathbb{Z}^2_{\text{even}}\}$ be the collection of the scaled paths.

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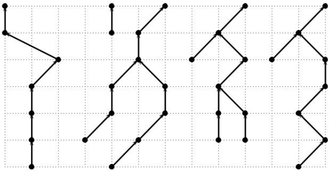

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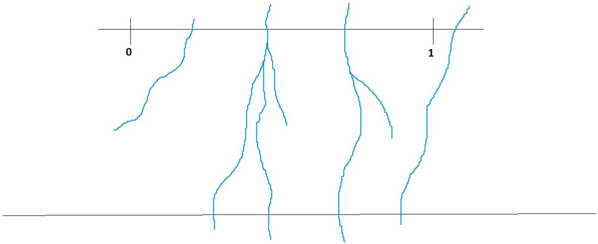
Idea of the proof: η_K

The difficulty is $\mathbb{P}\{L \geq n\} \to 0$ as $n \to \infty$.



For $K \subset \Pi$ we define a counting random variable as follows

$$\nu_K := \{\pi(0) : \sigma_\pi \leq -1 \text{ and } \pi(0) \in [0,1]\} \text{ and } \eta_K := \#\nu_k.$$

Idea of the proof: $\mathbb{E}(\eta_{\mathcal{X}_n})$

Using the translation invariance of our model, we have,

$$\begin{aligned}
\mathbb{E}(\eta_{\mathcal{X}_n}) &= \sum_{k=0}^{\lfloor\sqrt{n}\rfloor} \mathbb{E}(\mathbf{1}_{\{L(k,0)>n\}}) \\
&= \sqrt{n}\mathbb{E}(\mathbf{1}_{\{L>n\}}) \\
&= \sqrt{n}\mathbb{P}(L>n).
\end{aligned}$$

Idea of the proof: Jt. convergence to double Brownian web

Theorem (Roy, S., Sarkar(2015))

$(\mathcal{X}_n, \widehat{\mathcal{X}}_n) \Rightarrow (\mathcal{W}, \widehat{\mathcal{W}})$ *as* $n \to \infty$

Idea of the proof: Convergence of $\mathbb{E}(\eta_{\mathcal{X}_n})$

Using the earlier result we further show that

Theorem (Roy, S., Sarkar(2015))

$\mathbb{E}(\eta_{\mathcal{X}_n}) \to \mathbb{E}(\eta_{\mathcal{W}}) = 1/\sqrt{\pi}$ *as* $n \to \infty$

which gives us that $\sqrt{n}\mathbb{P}(L \geq n) \to 1/\sqrt{\pi}$.
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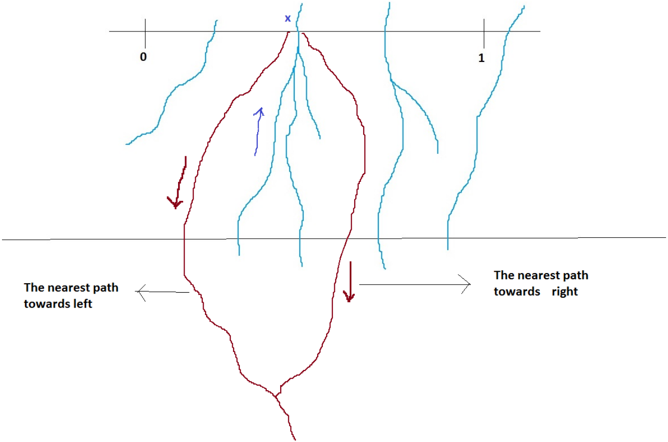
This idea can be extended further.

Idea of the proof: Other random variables associated to $\nu_{K\times\widehat{K}}$

For $(K,\widehat{K})$ with $K\subset\Pi$, $\widehat{K}\subset\widehat{\Pi}$ and for $x\in\nu_{K\times\widehat{K}}$ we consider



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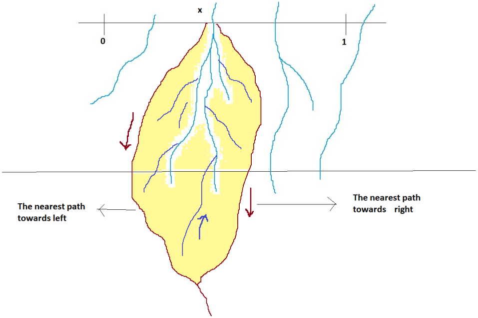
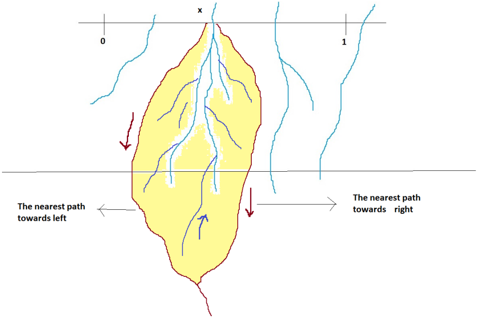


- Under suitable assumptions on $(K,\widehat{K})$, this naturally gives us a tree-like metric space.
- Consider completion of this metric space denoted by $\mathcal{M}^{(x,0)} = \mathcal{M}^{(x,0)}(K\times\widehat{K})$.

Idea of the proof: Other random variables associated to $\nu_{K\times\widehat{K}}$



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Idea of the proof: Other random variables associated to $\nu_{K\times\widehat{K}}$

For f, a bounded continuous real valued function on the *universal* metric space, we define

$$\kappa_{(K,\widehat{K})}(f) := \begin{cases} 0 & \text{if } \nu_K \cap [0,1] = \emptyset \\ \sum_{x\in\nu_K\cap[0,1]} f(I(\mathcal{M}^{(x,0)})) & \text{otherwise.} \end{cases}$$

We consider $\kappa(f) := \kappa_{(\mathcal{W},\widehat{\mathcal{W}})}(f)$ and $\kappa_n(f) := \kappa_{(\mathcal{X}_n,\widehat{\mathcal{X}}_n)}(f)$ and show that

$$\mathbb{E}(\kappa_n(f)) \to \mathbb{E}(\kappa_n(f)) \text{ as } n \to \infty.$$

Idea of the proof: Convergence of $\mathbb{E}(\kappa_n(f))$

To achieve convergence in expectation we require

- convergence in distribution
- uniform boundedness of the sequence (standard argument)

To achieve weak convergence we show that for $\nu_{\mathcal{X}_n} = \{x_n\}$ and $\nu_{\mathcal{W}} = \{x\}$

$$d_{\mathrm{GH}}(\mathcal{M}^{(x_n,0)}, \mathcal{M}^{(x,0)}) \to 0 \text{ almost surely as } n \to \infty$$

(on some probability space).

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Idea of the proof: GH distance calculation

For T_1, T_2 a correspondence $\mathcal{R} \subseteq T_1 \times T_2$ is such that for any $x_1 \in T_1$ there exists $(x_1, x_2) \in \mathcal{R}$ with $x_2 \in T_2$ and for any $y_2 \in T_2$ there exists $(y_1, y_2) \in \mathcal{R}$ with $y_1 \in T_1$.
For any correspondence $\mathcal{R}$ we define

$$\mathrm{dis}(\mathcal{R}) := \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

Finally

$$d_{\mathrm{GH}}(T_1, T_2) := 1/2 \inf_{\mathcal{R} \in C((T_1, T_2), (\rho_1, \rho_2))} \mathrm{dis}(\mathcal{R}).$$

where $C((T_1, T_2), (\rho_1, \rho_2))$ denote the set of *all* correspondences between rooted trees (T_1, ρ_1) and (T_2, ρ_2).

Wedge convergence

(System of coalescing (scaled)paths together with their coalescing times)
$\Rightarrow$ (System of coalescing Brownian motions together with their coalescing times)

More precisely we show that convergence happens in $d_{\mathcal{H}\times\widehat{\mathcal{H}}}(\cdot) + d'(\cdot)$ where for K and K' compact subsets of Π

$$d'(K,K') := \sup_{\pi_1,\pi_2\in K}\inf_{\pi'_1,\pi'_2\in K'}\{|\sigma_{\pi_1}-\sigma_{\pi'_1}|\vee|\sigma_{\pi_2}-\sigma_{\pi'_2}|\vee|t_{1,2}-t'_{1,2}|\}\vee \sup_{\pi'_1,\pi'_2\in K'}\inf_{\pi_1,\pi_2\in K}\{|\sigma_{\pi_1}-\sigma_{\pi'_1}|\vee|\sigma_{\pi_2}-\sigma_{\pi'_2}|\vee|t_{1,2}-t'_{1,2}|\}.$$

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$$d'(K,K') := \sup_{\pi_1,\pi_2\in K} \inf_{\pi_1',\pi_2'\in K'} \{|\sigma_{\pi_1} - \sigma_{\pi_1'}| \vee |\sigma_{\pi_2} - \sigma_{\pi_2'}| \vee |t_{1,2} - t_{1,2}'|\} \vee \sup_{\pi_1',\pi_2'\in K'} \inf_{\pi_1,\pi_2\in K} \{|\sigma_{\pi_1} - \sigma_{\pi_1'}| \vee |\sigma_{\pi_2} - \sigma_{\pi_2'}| \vee |t_{1,2} - t_{1,2}'|\}.$$

Idea of the proof:

The main idea is to show that if convergence does not happen in this metric then for some sub sequential limit of $\{\mathcal{X}_n, \widehat{\mathcal{X}}_n : n \in \mathbb{N}\}$ some forward path spend positive Lebesgue time with some backward path, which gives a contradiction.

Universality of the scaling limit

Our proof essentially based on joint convergence to the Brownian web and its dual. Hence it is an universality class type of result as we have the same limit holds for all *drainage network model with non-crossing paths in the basin of attraction of the Brownian web*.

Open problems

(a) How does the excursion path for $\mathcal{T}_{\mathcal{W}}$ is distributed?

(b) Can we say that (*on a coupled space*) $\mathcal{T}_{\mathcal{W}}$ almost surely determines $\widehat{\mathcal{T}}_{\mathcal{W}}$?

(c) What is the Minkowski dimension of $\mathcal{T}_{\mathcal{W}}$ and $\widehat{\mathcal{T}}_{\mathcal{W}}$?

(d) Does Tokunaga relation holds for $\mathcal{T}_{\mathcal{W}}$ and $\widehat{\mathcal{T}}_{\mathcal{W}}$?

Thank you