TOPOLOGY OF RANDOM POINTS

YOGESHWARAN. D.

MARCH 2017, IISC.

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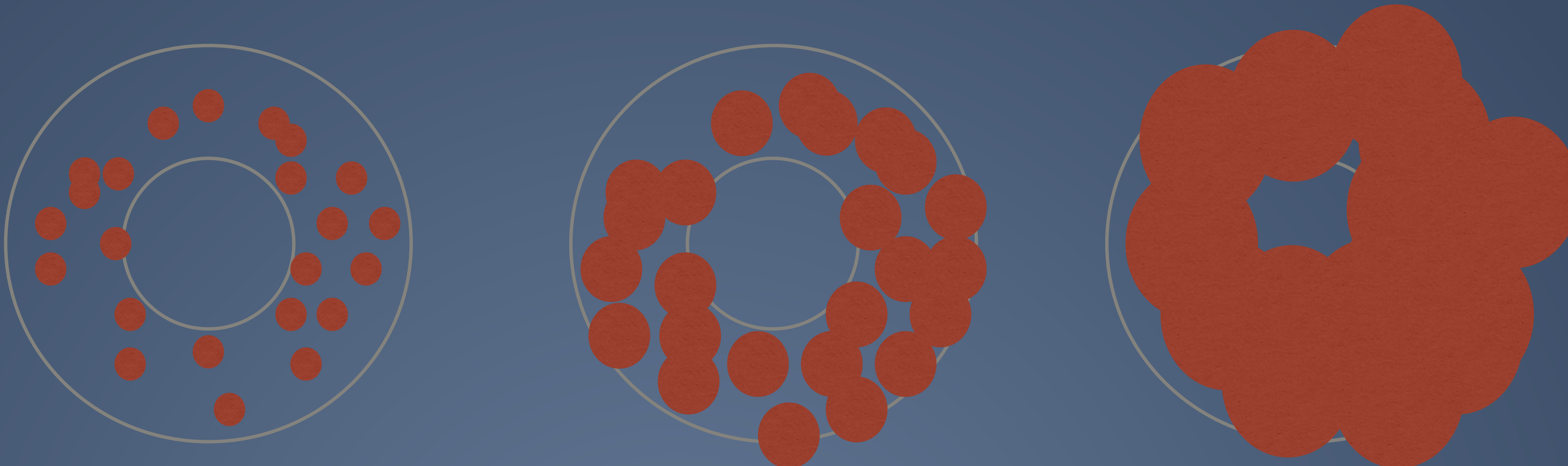

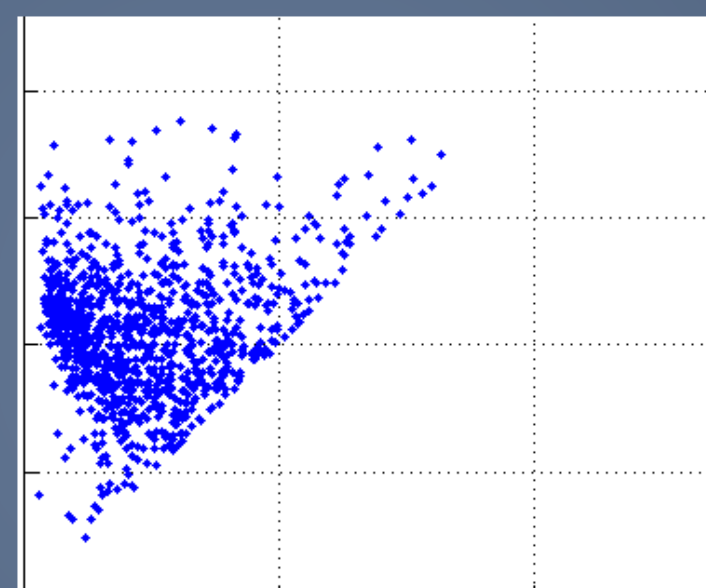

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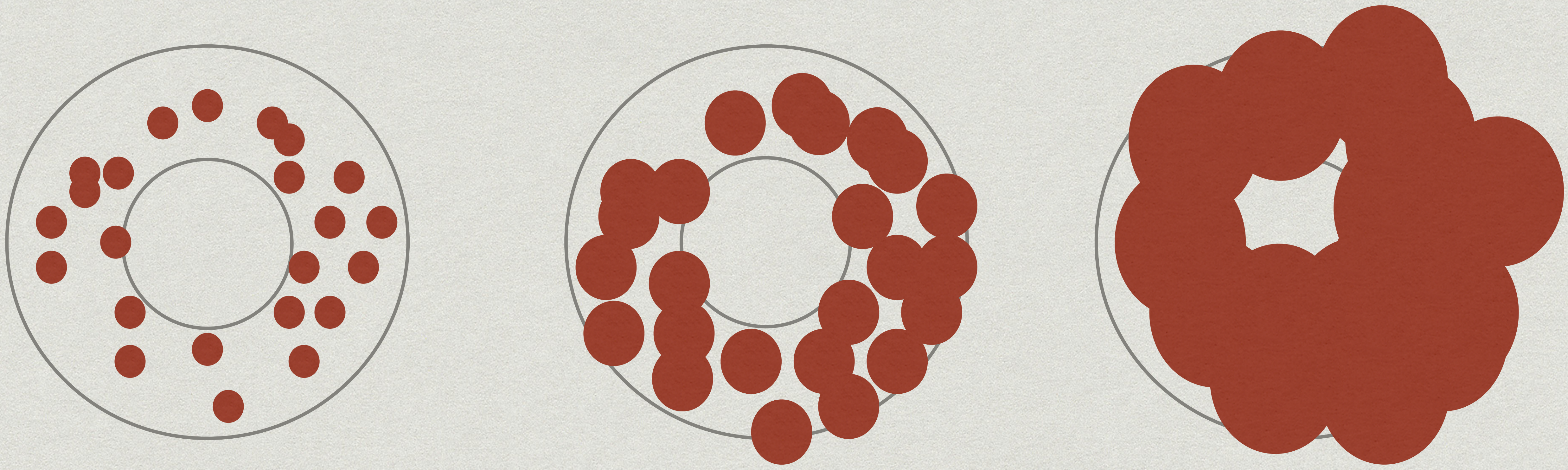

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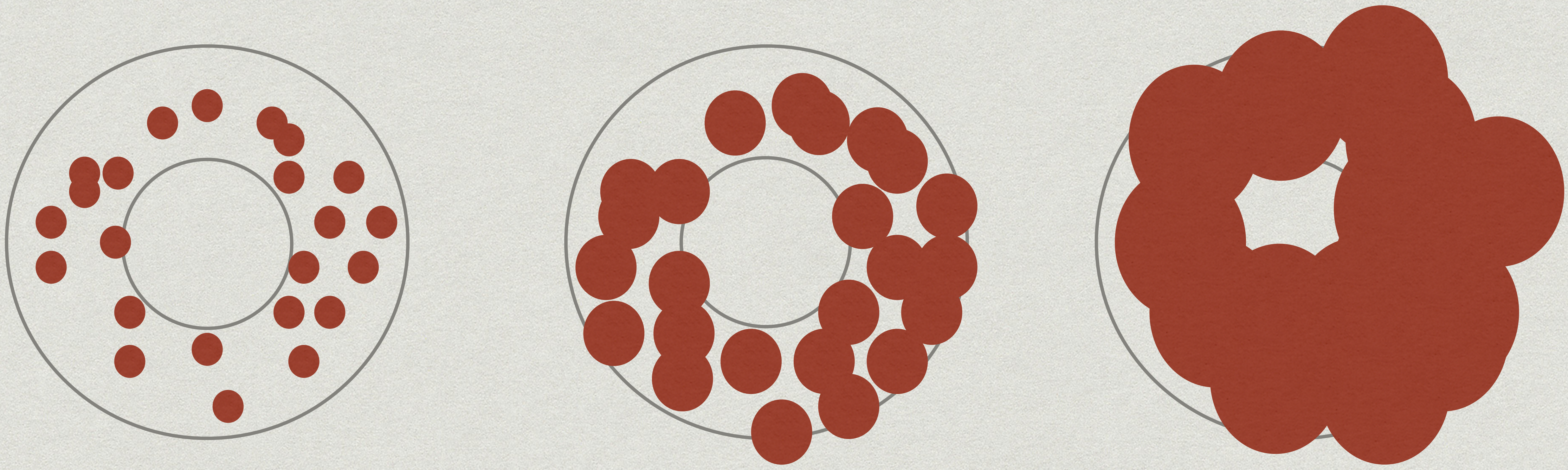

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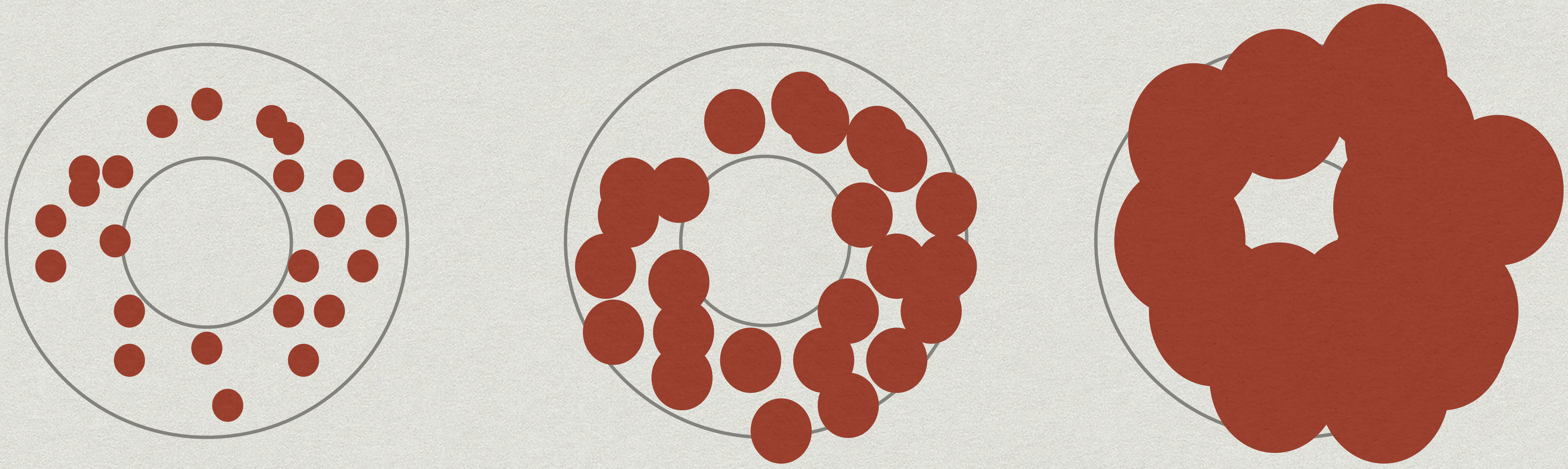

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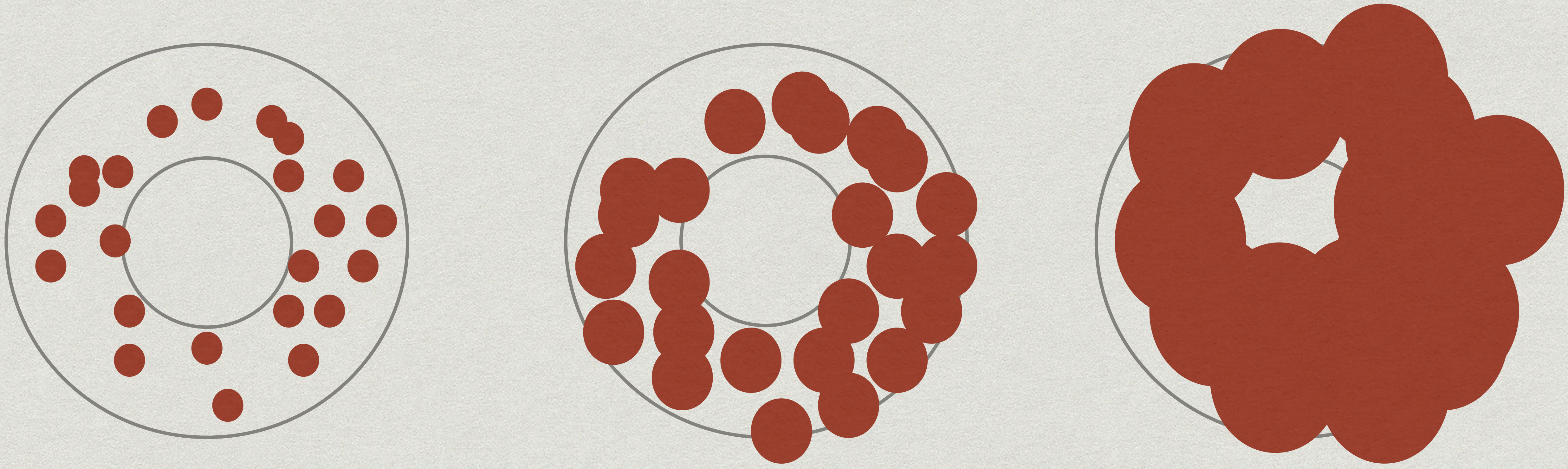

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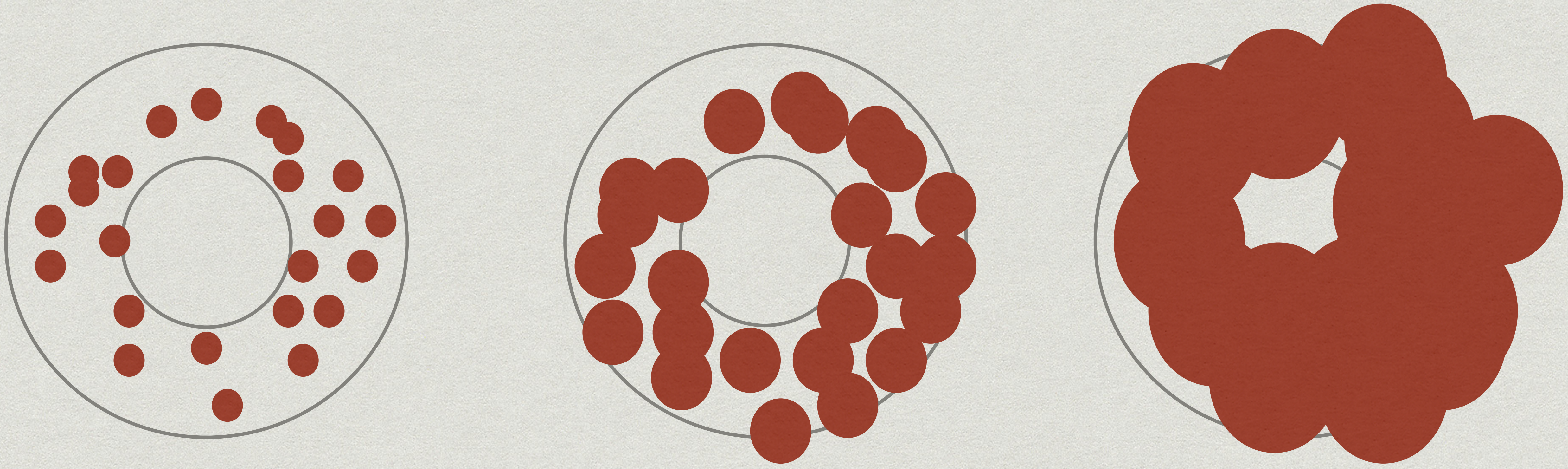

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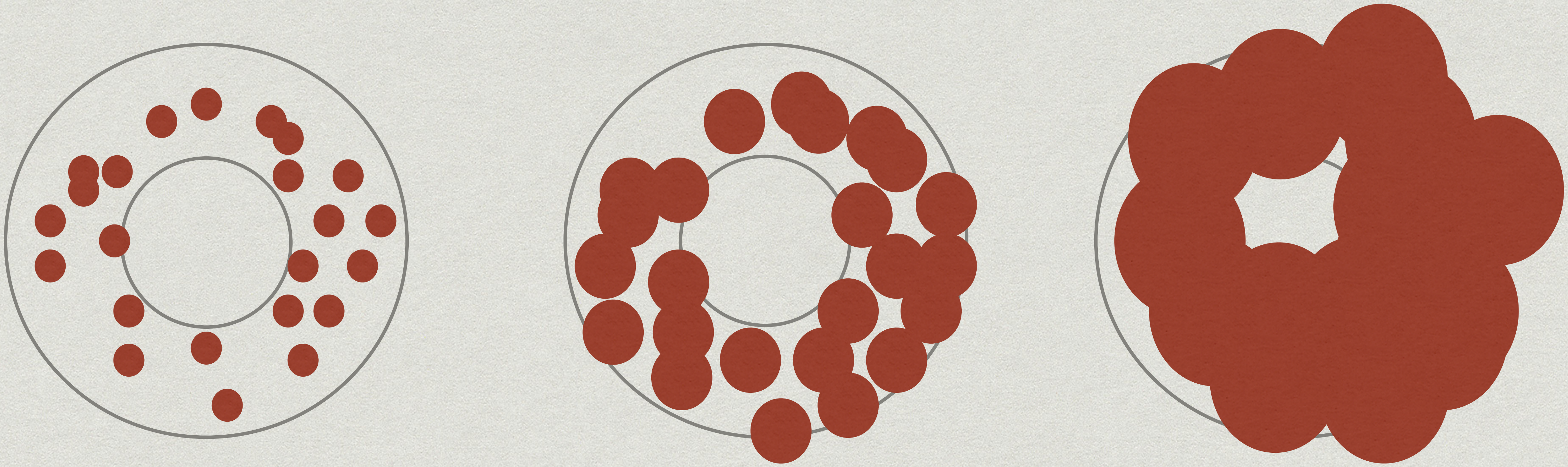

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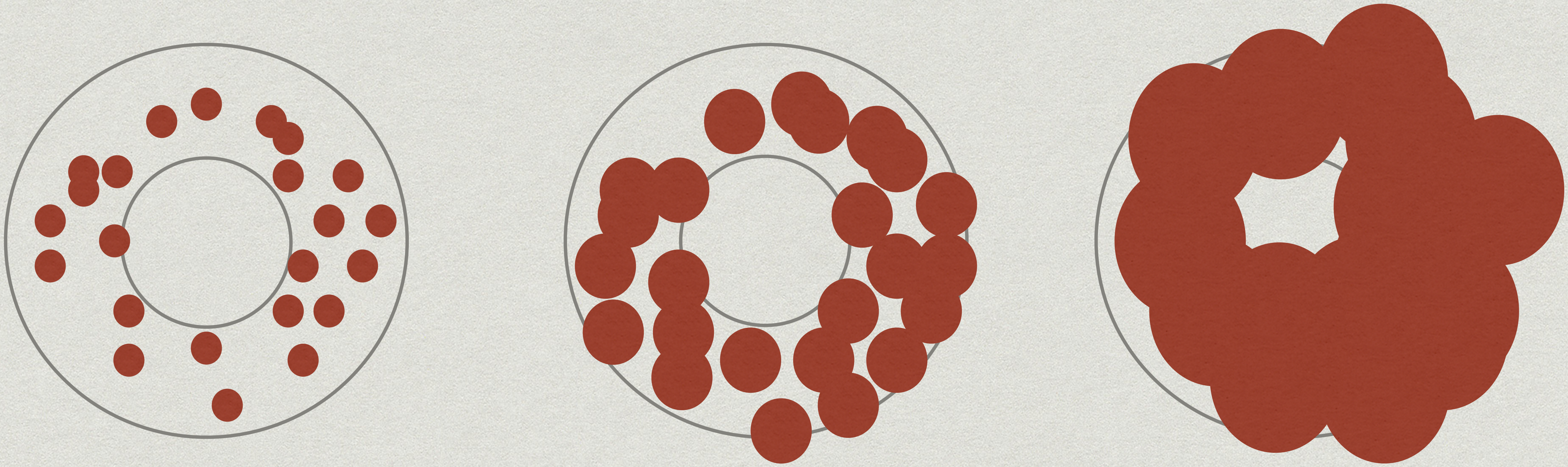

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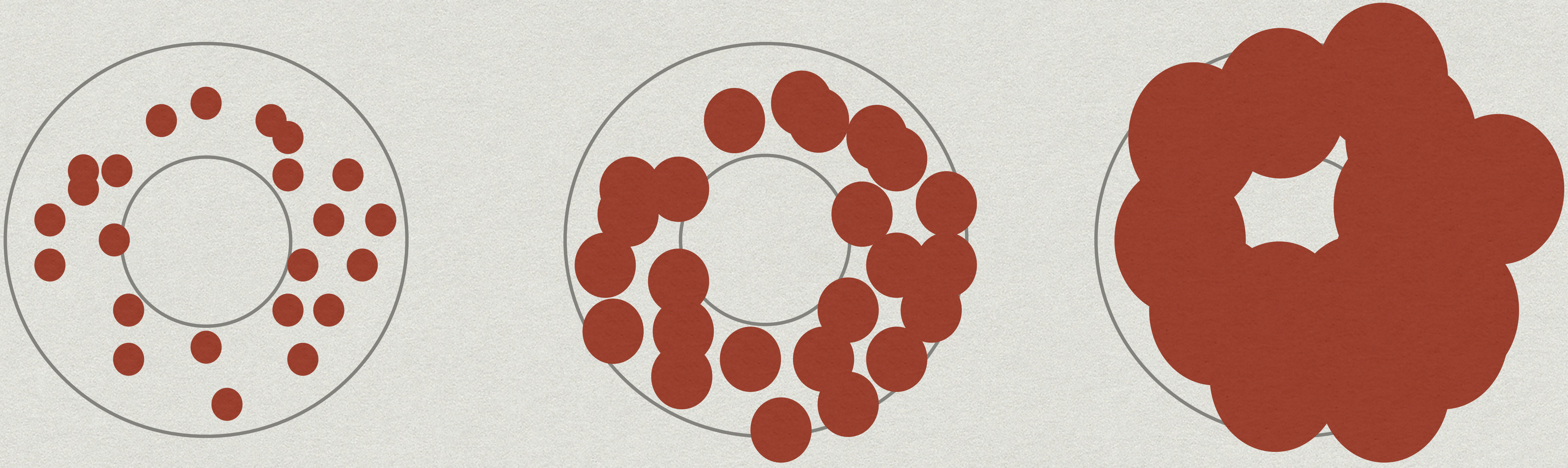

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Other topological summaries ?

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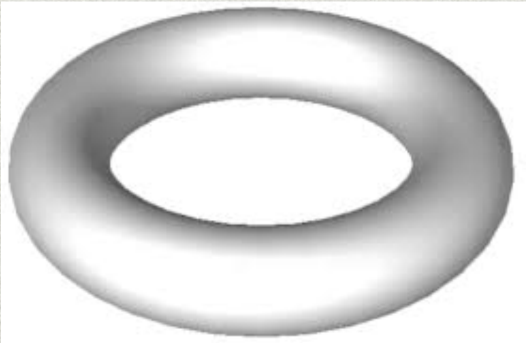

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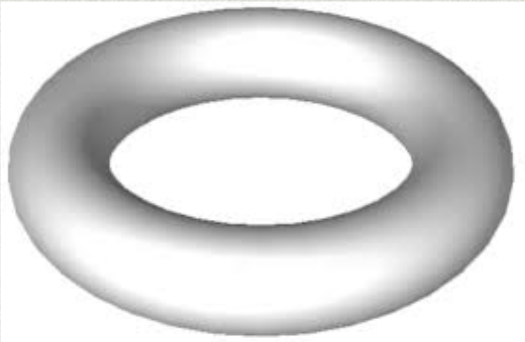

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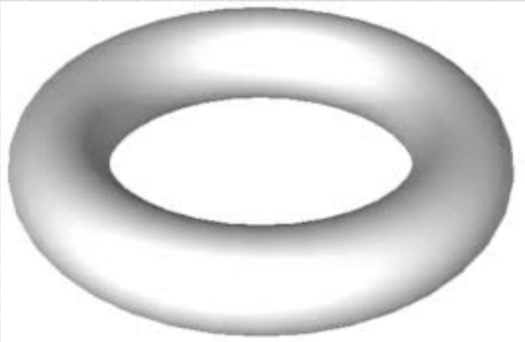

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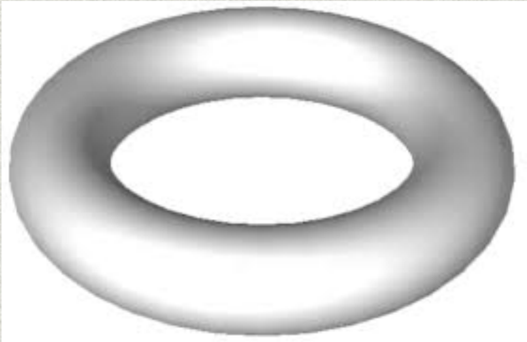

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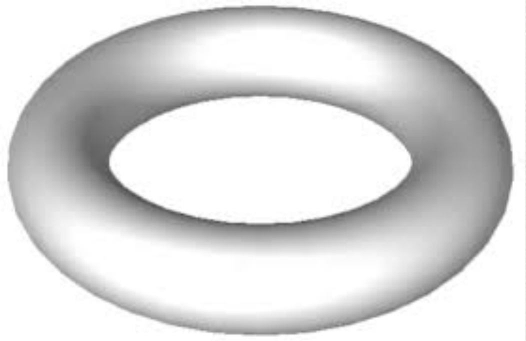

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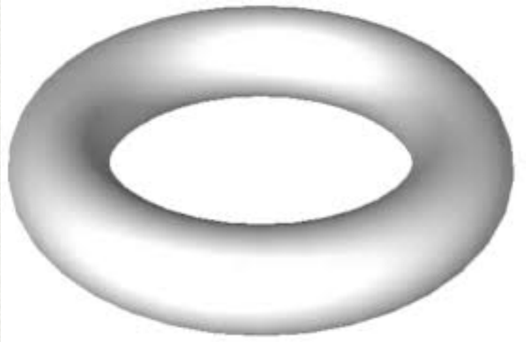

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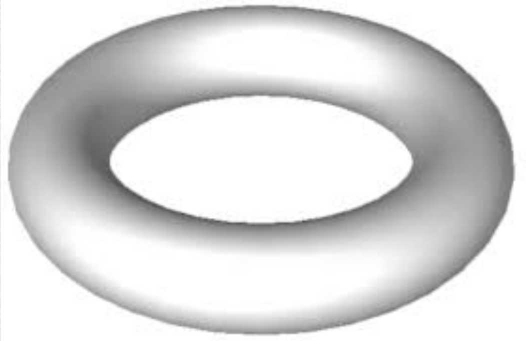

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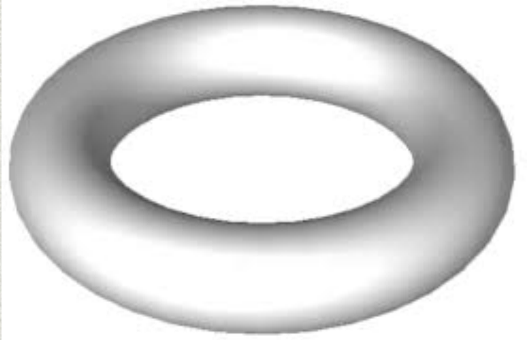

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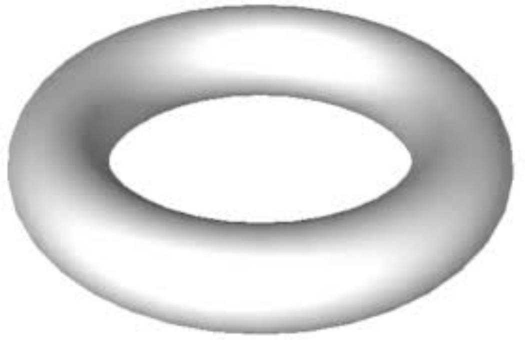

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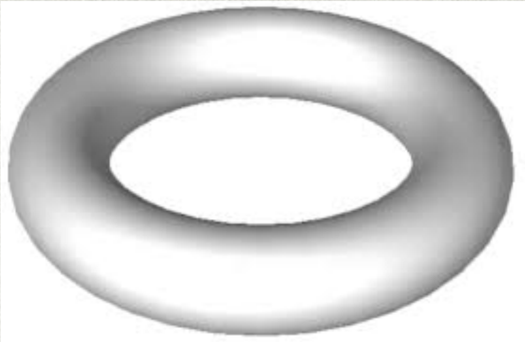

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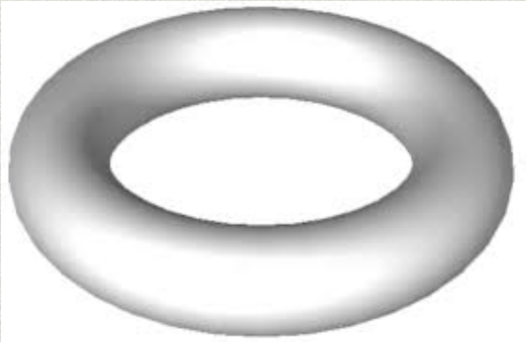

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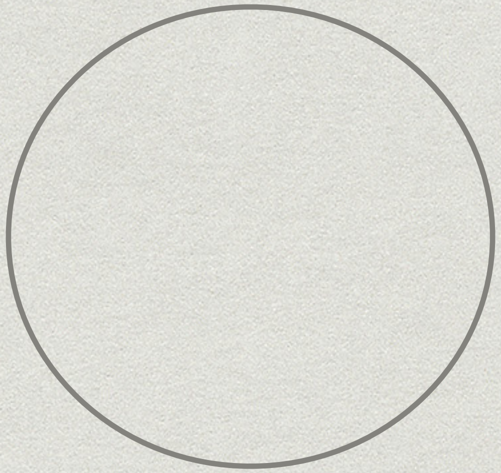

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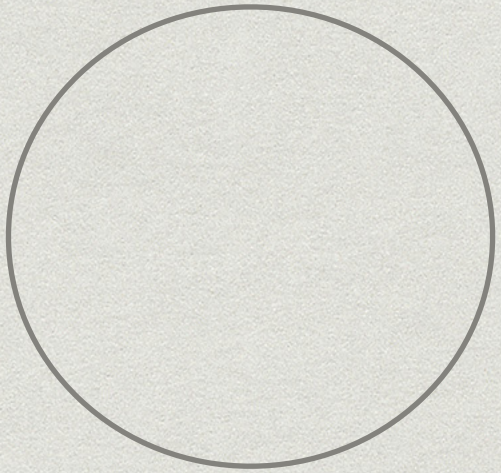

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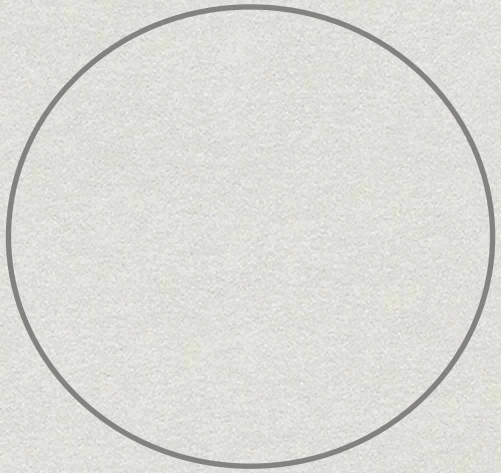

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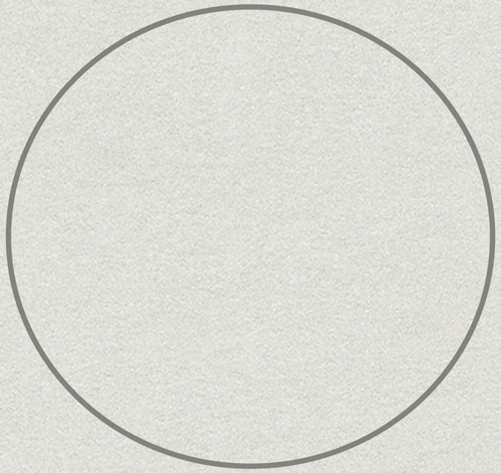

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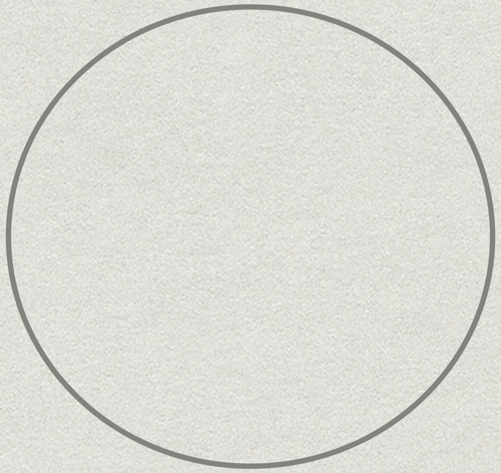

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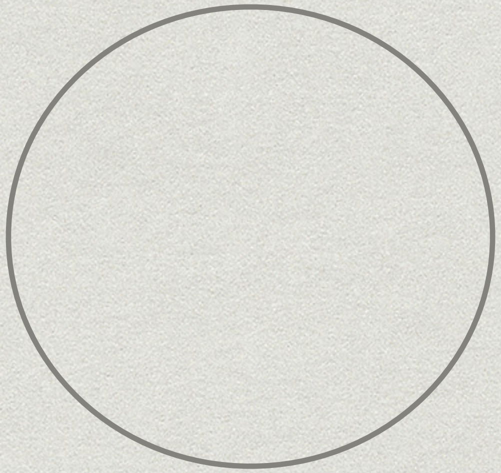

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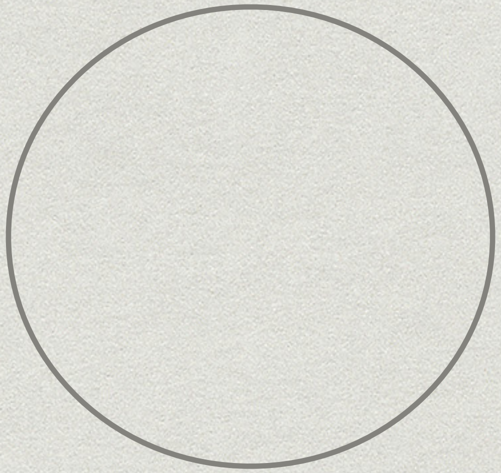

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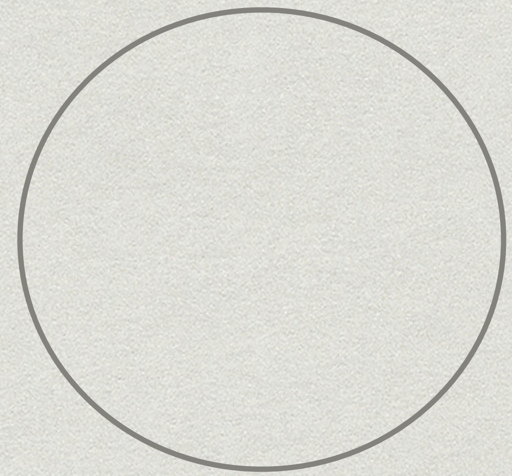

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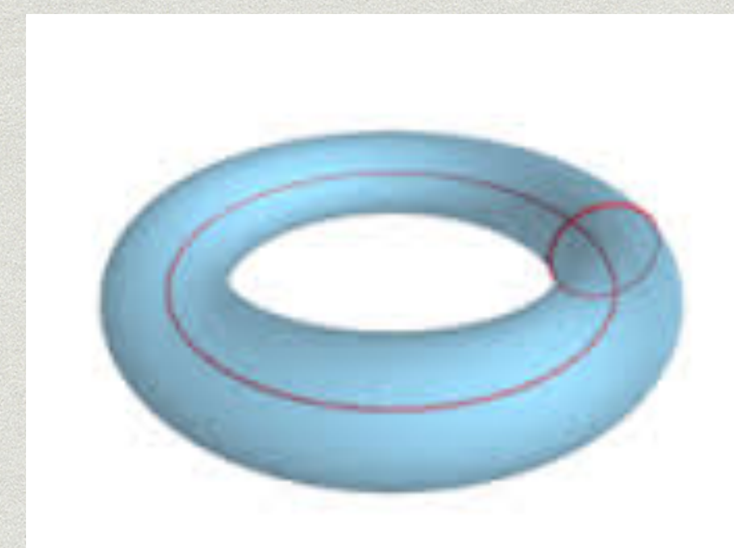

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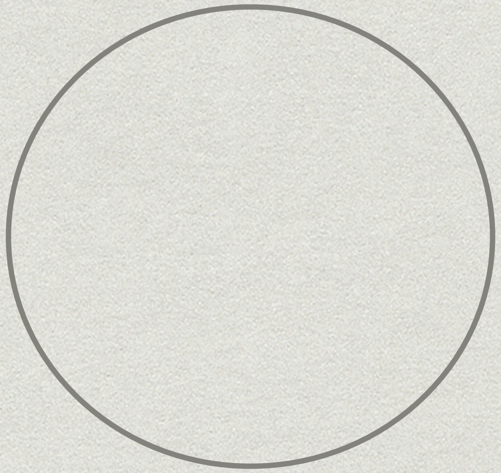

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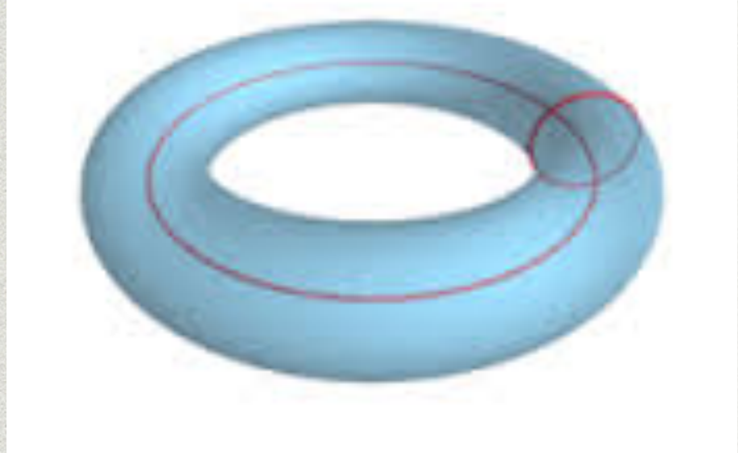

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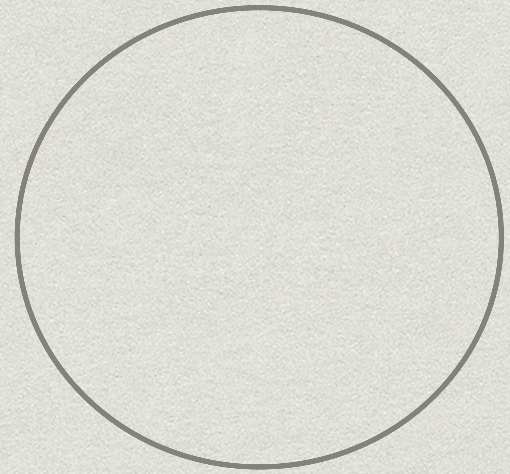

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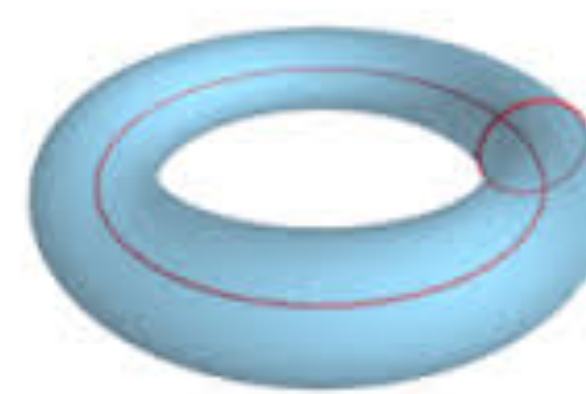

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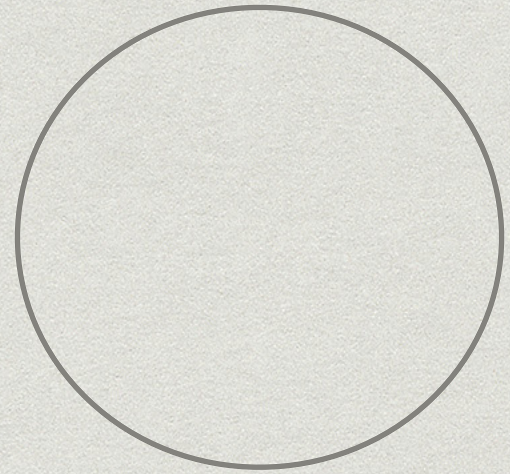

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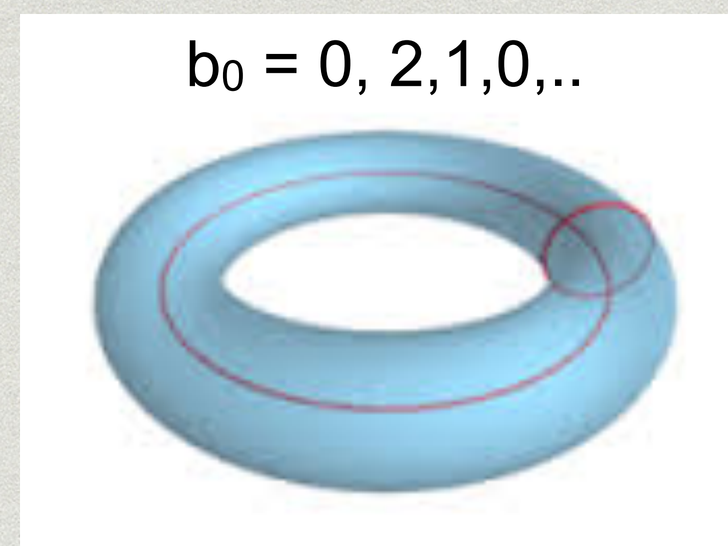

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0-dim result - **M. D. Penrose (1999),**
General k-result - **O. Bobrowski - S. Weinberger (2015),**
d-dim result - **L. Flatto - D. J. Newman (1977), P. Hall (1986).**

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Our conjecture for β_1 : $n\theta_d r_n^d = \log n - \log\log n$.

Combinatorial vs Topological Connectivity

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For $\epsilon \in (0,1)$

$$\mathbb{P}(\beta_k(C_n(r_n)) = \binom{d}{k}) \to \begin{cases} 0 & \text{if } \; n\theta_d r_n^d = (1-\epsilon)\log n \\ 1 & \text{if } \; n\theta_d r_n^d = (1+\epsilon)\log n \end{cases}$$

Combinatorial vs Topological Connectivity

Bobrowski - Weinberger, 2015.

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S.K. Iyer - D.Y., 2017+

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Why should we expect log n ?

Is the threshold for homotopy equivalence sharp ?

Can we infer the topology partially earlier OR
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Other topological summaries ?

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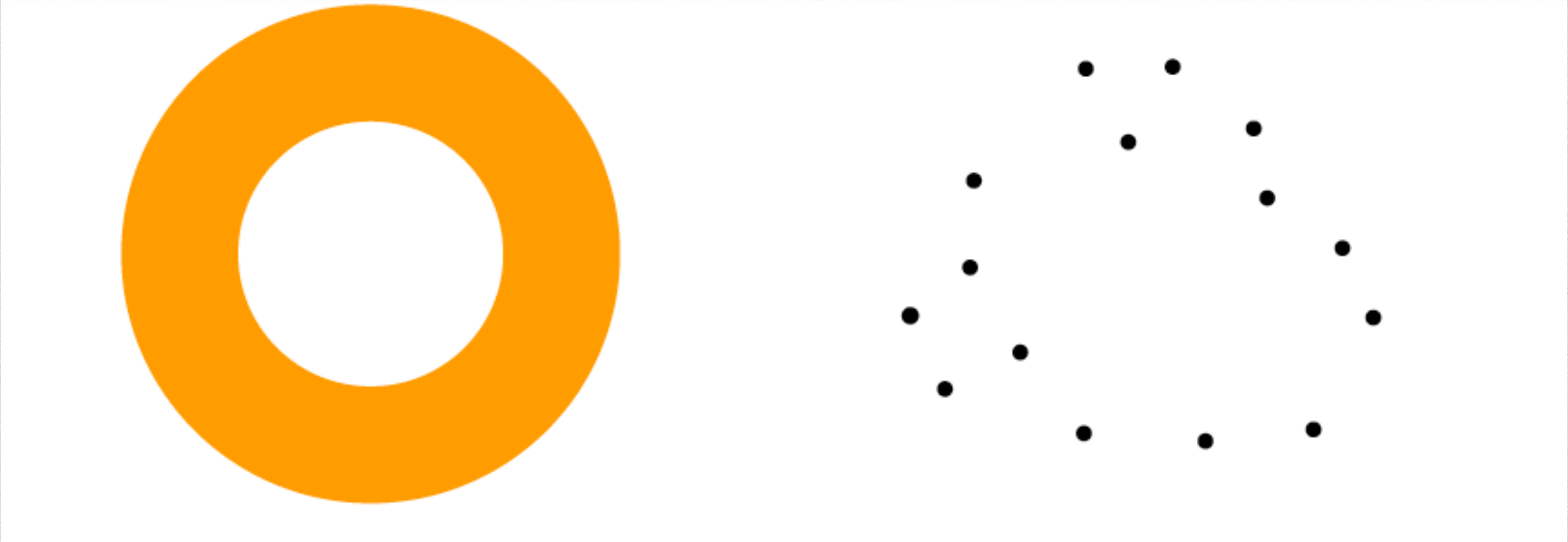
"We should seek out unfamiliar summaries of observational material, and establish their useful properties."

- **John W. Tukey,** "The future of data analysis", Ann. Math. Stat., 1962.

Evolution of Topology

Figures due to Gugan Thoppe

Evolution of Topology



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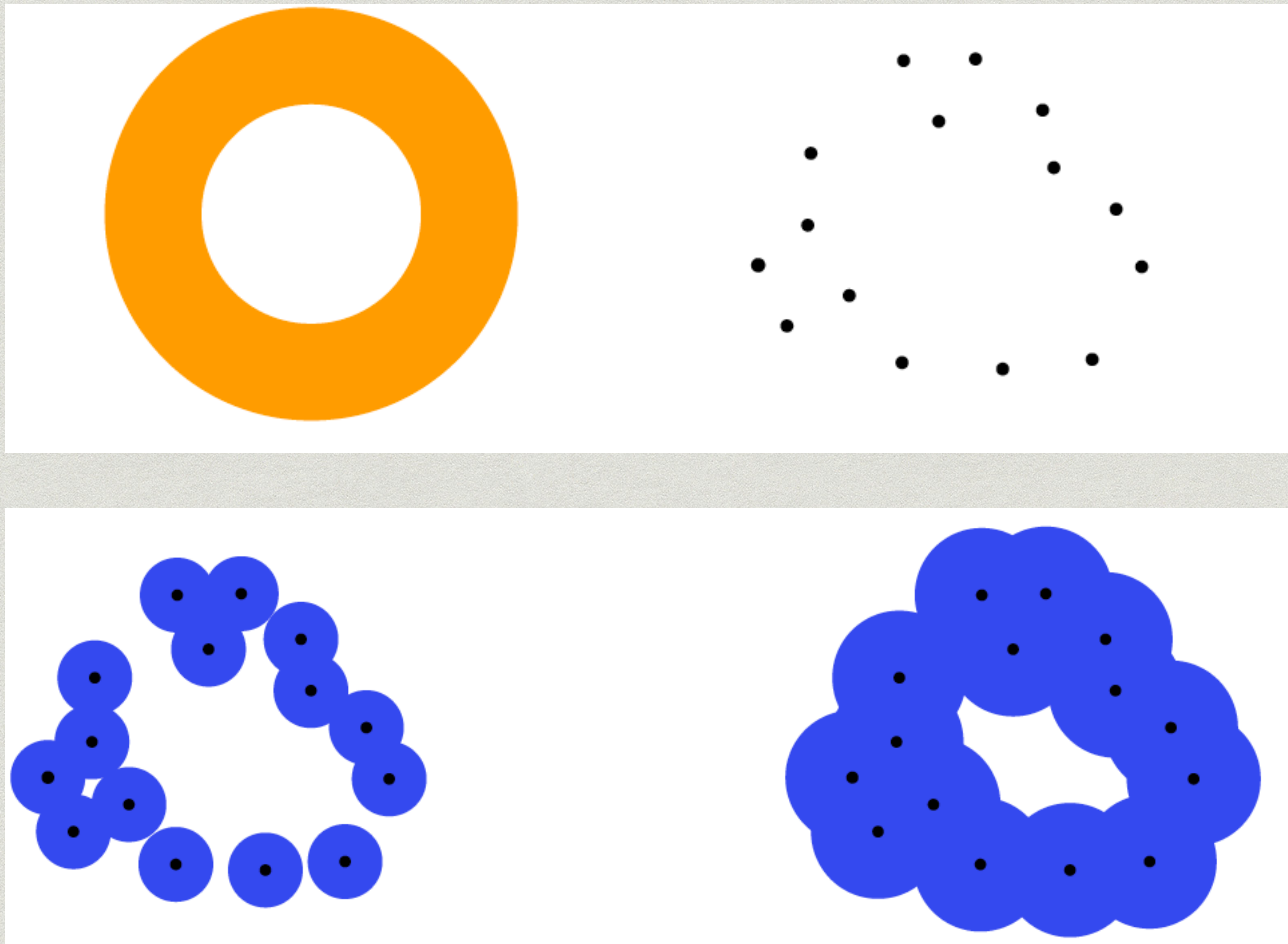
Evolution of Topology



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PERSISTENT HOMOLOGY / DIAGRAMS

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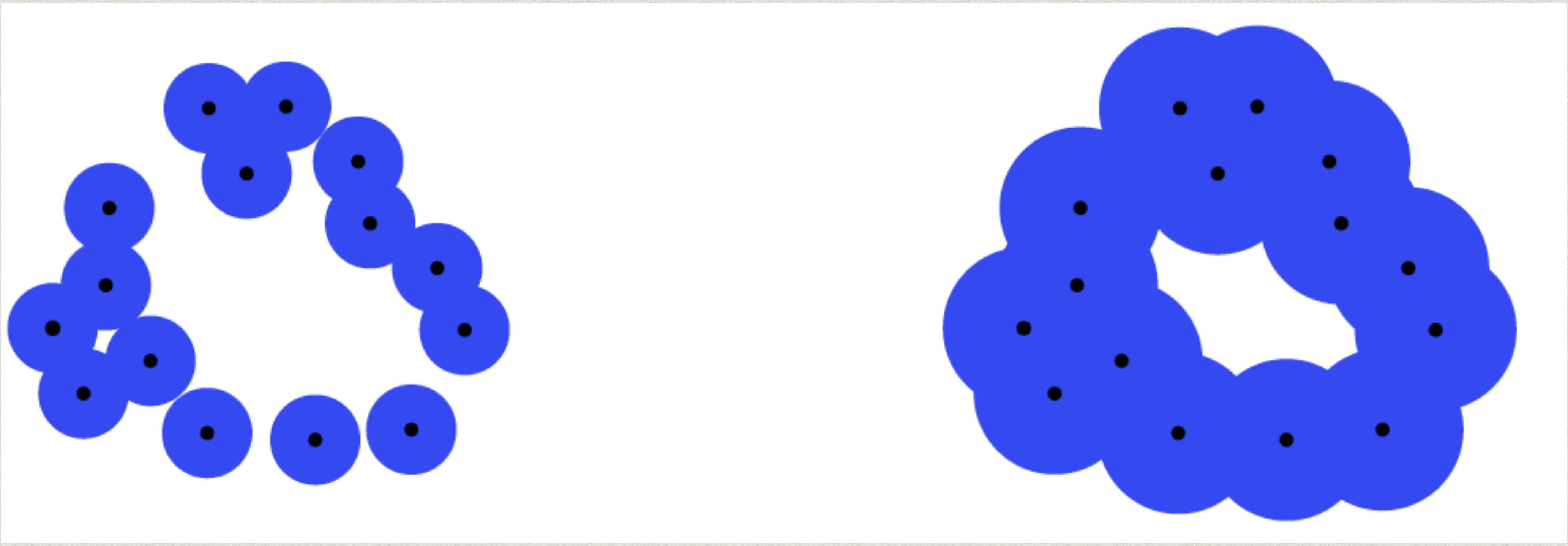

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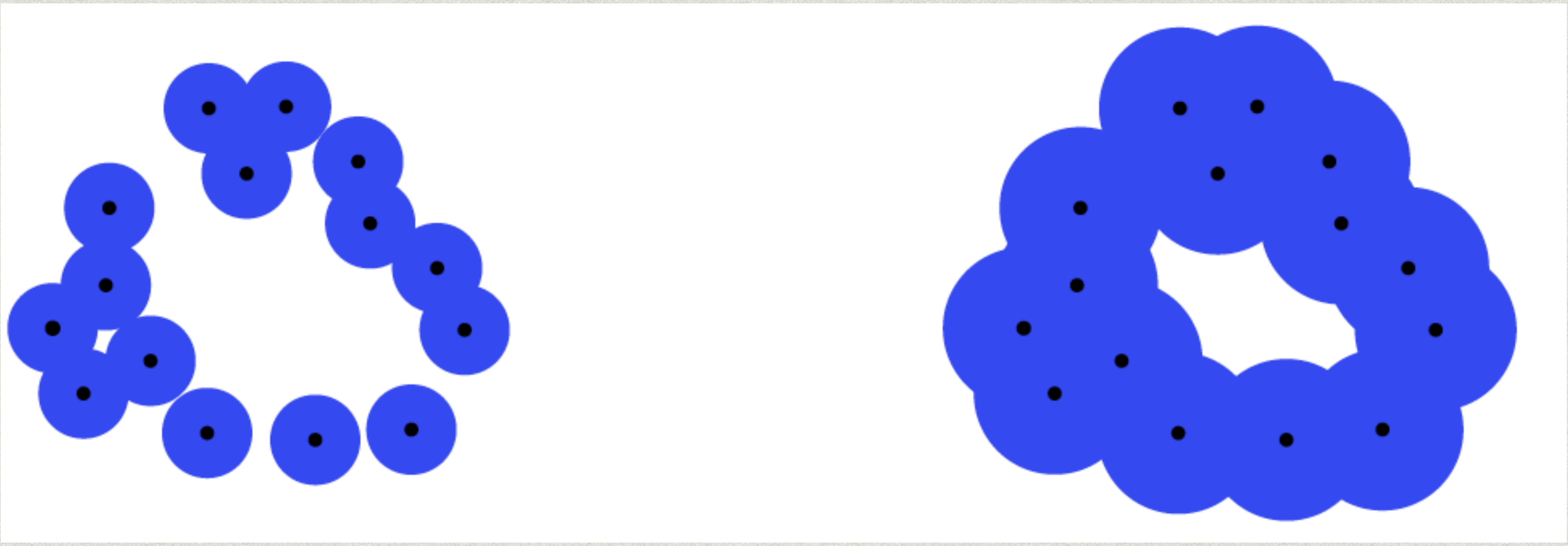

H_0 Persistence diagram

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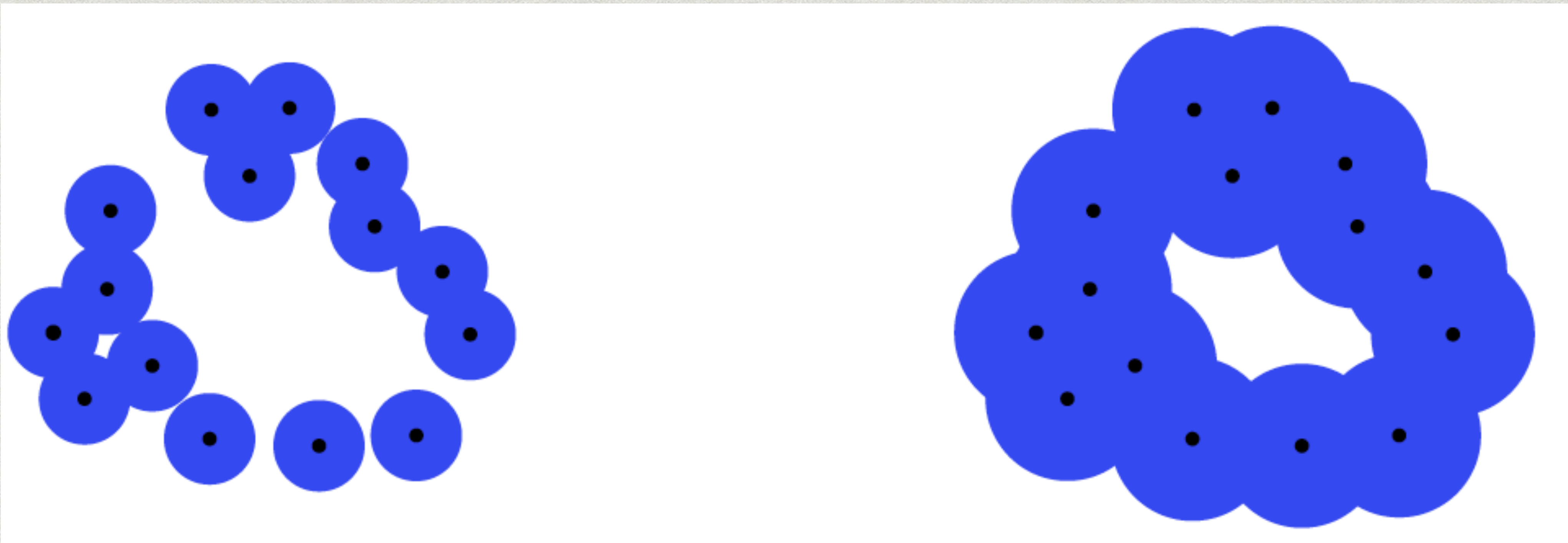

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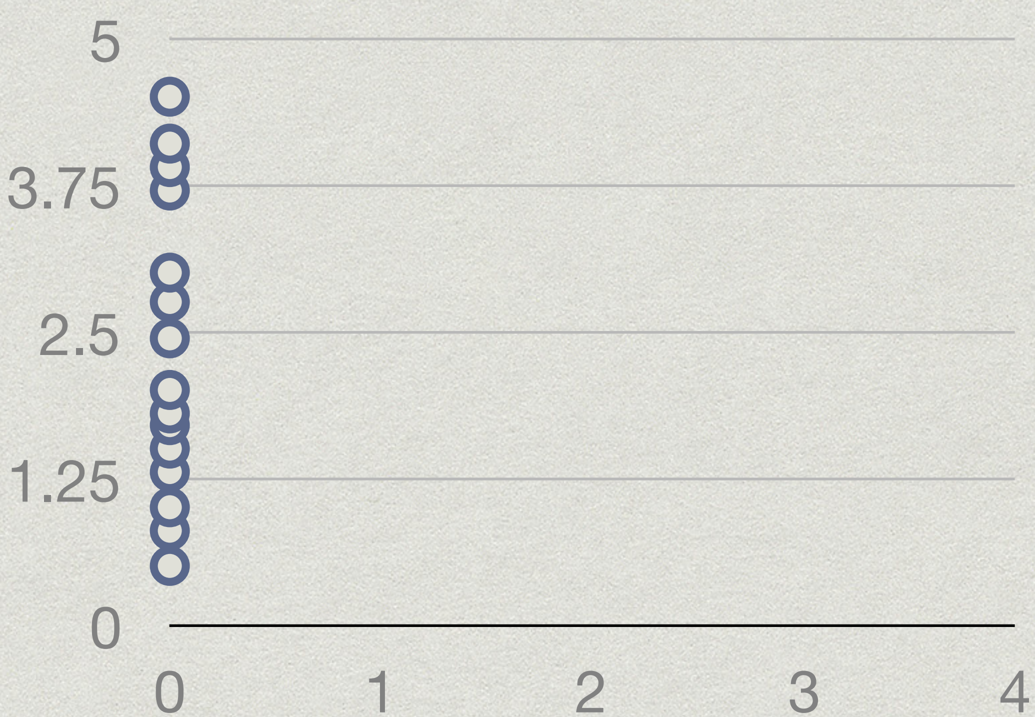

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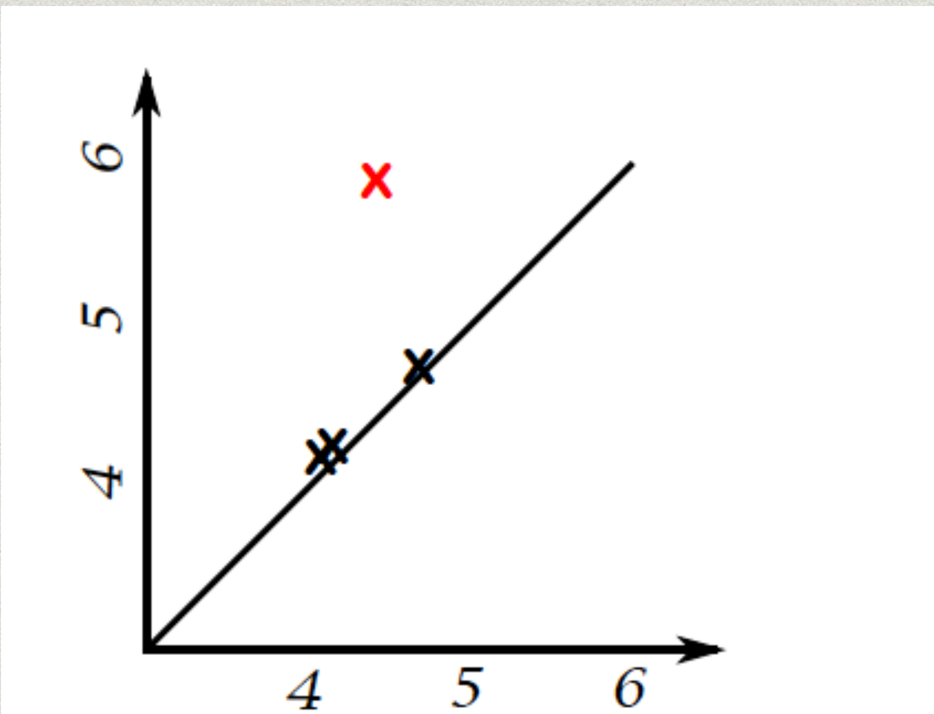

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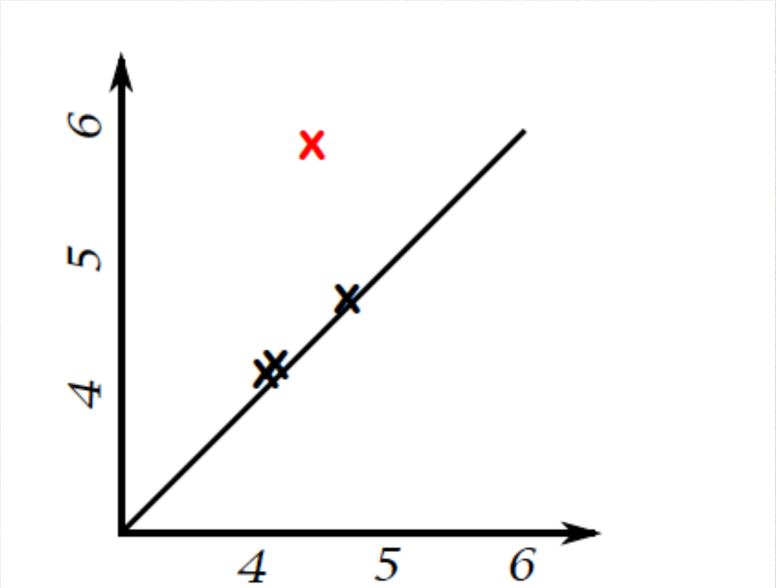

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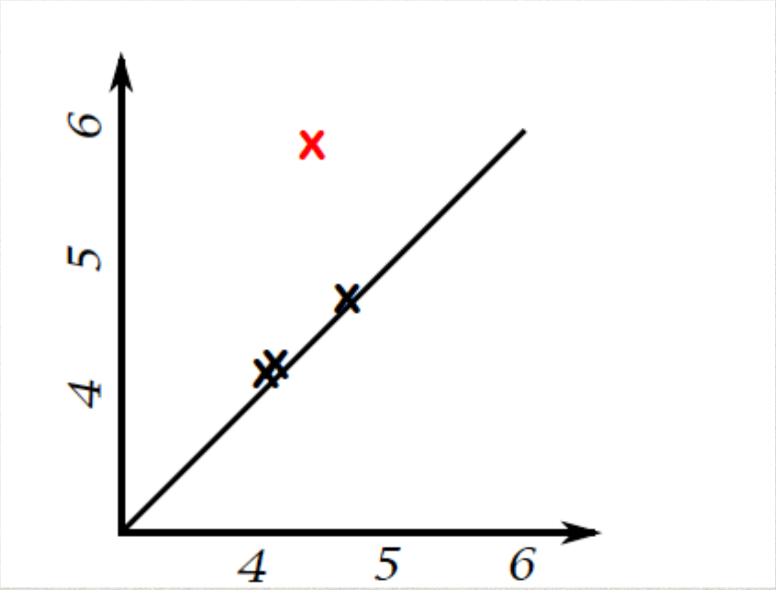

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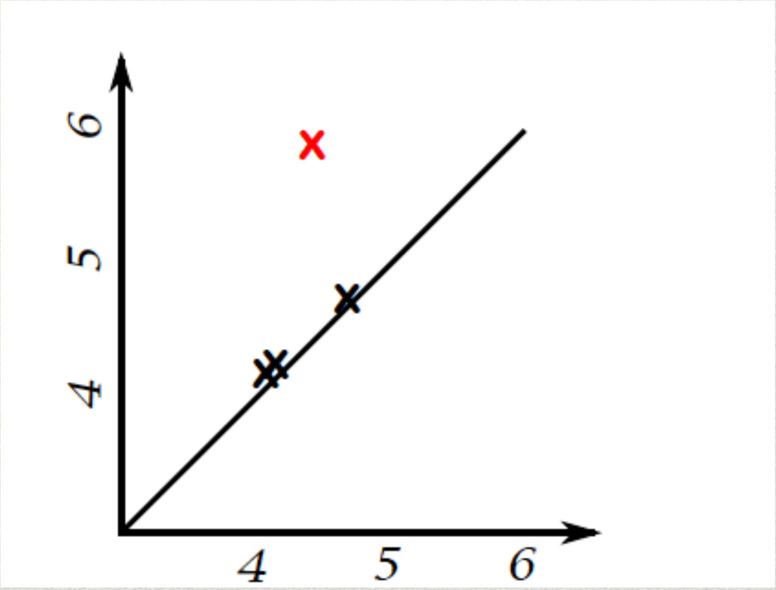

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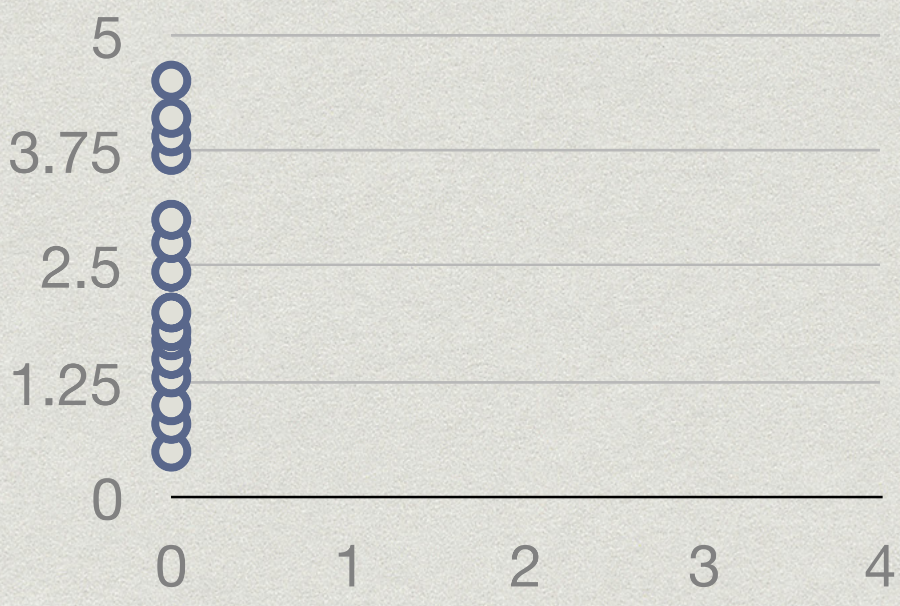

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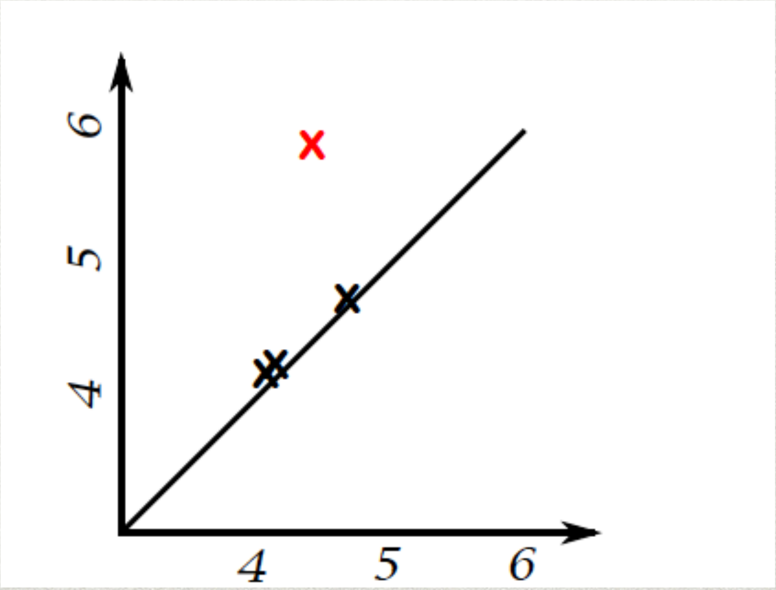

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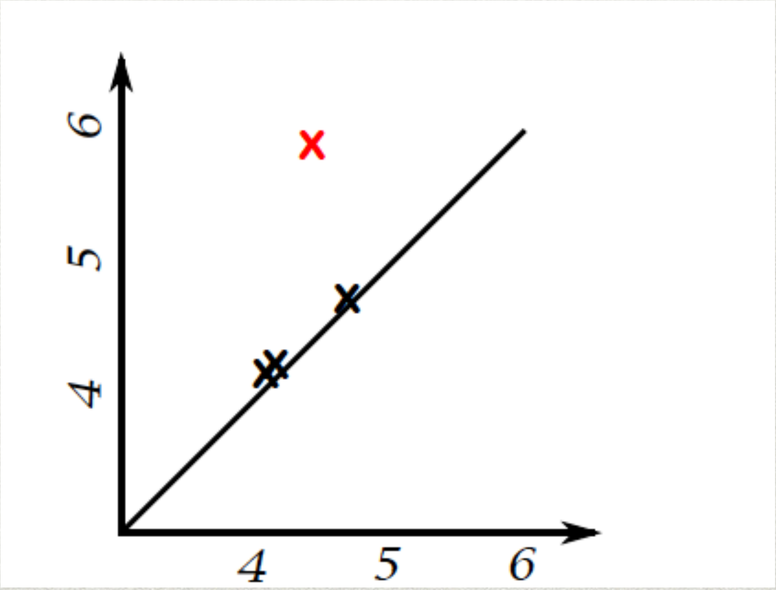

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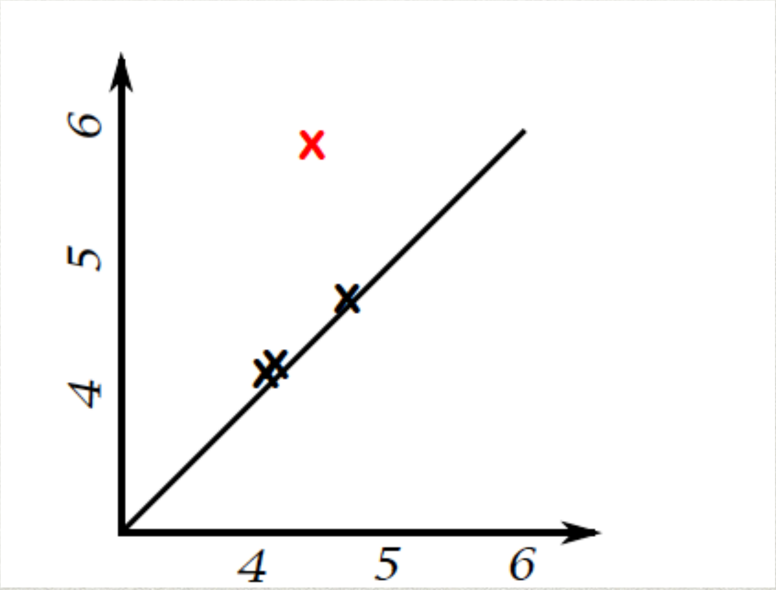

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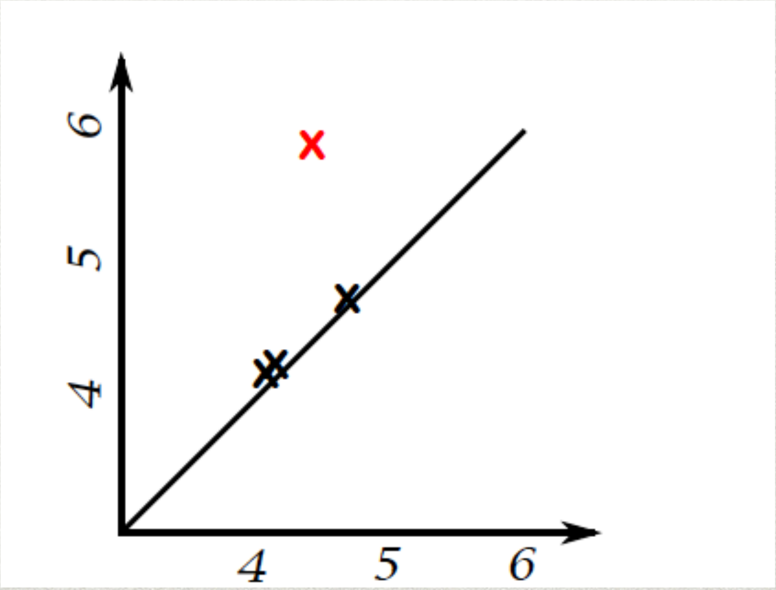

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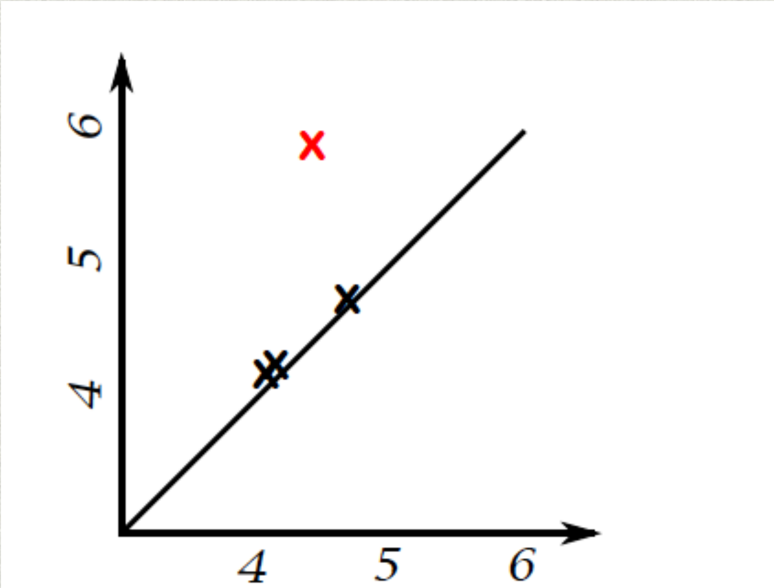

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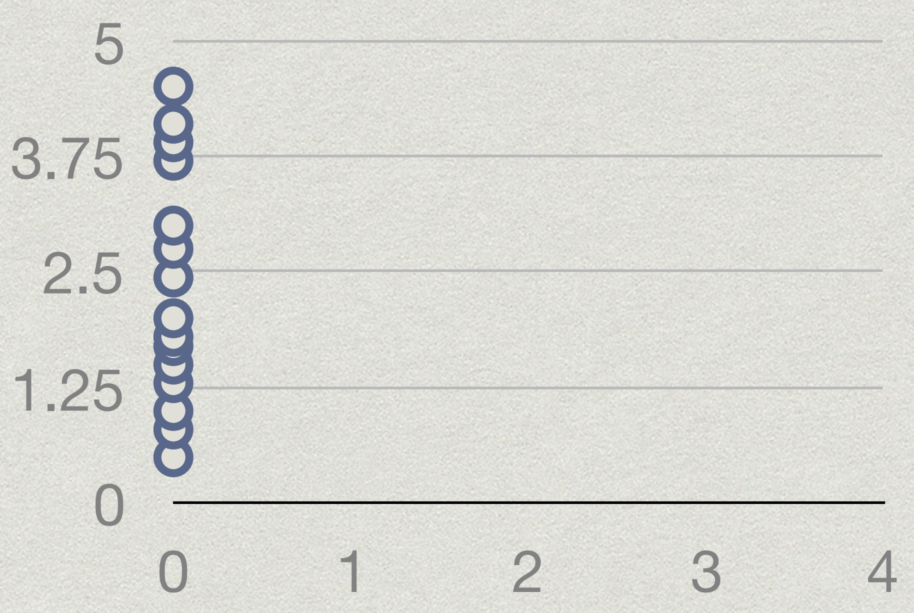

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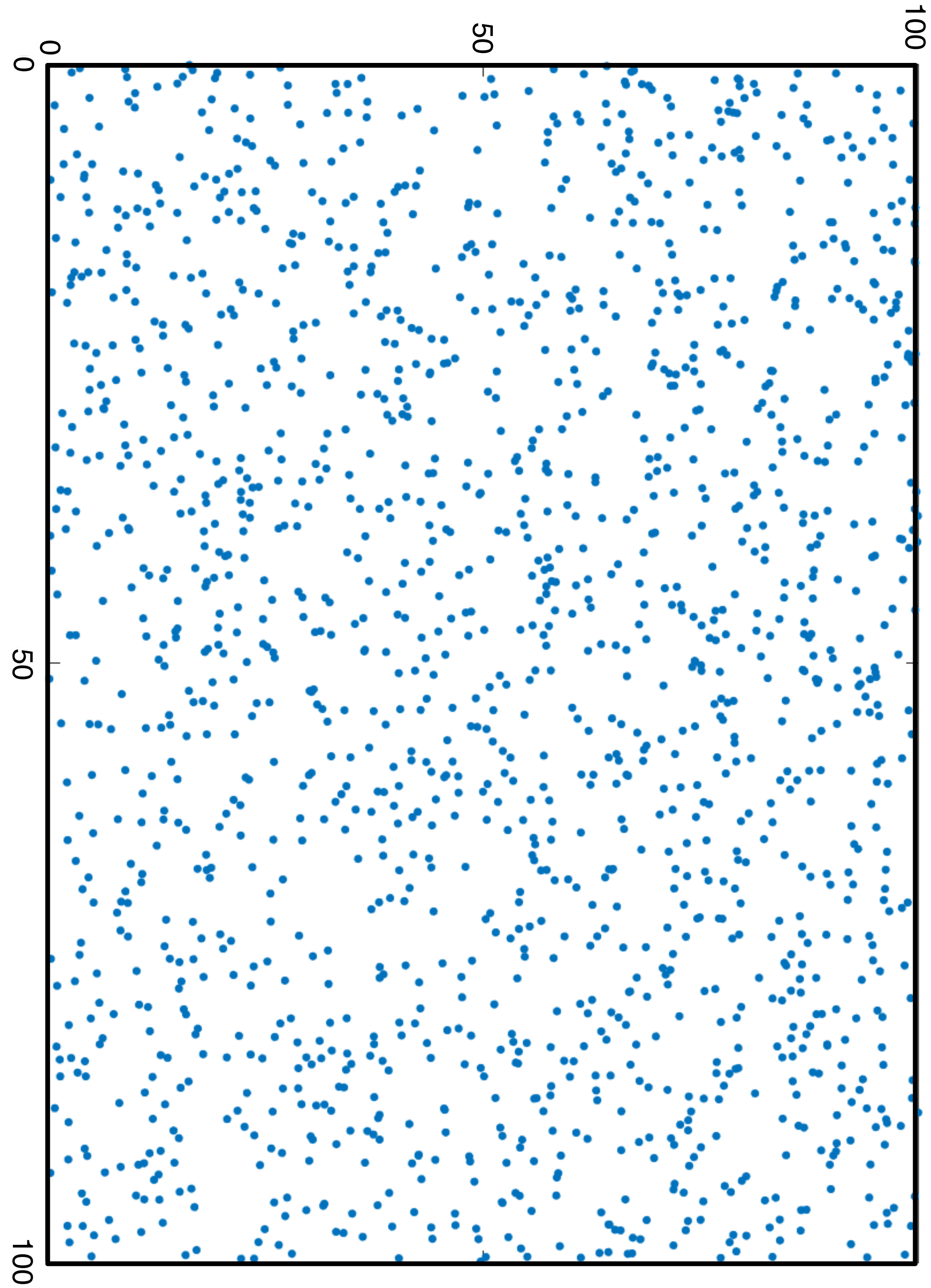
Figures from Duy, Hiraoka and Shirai.



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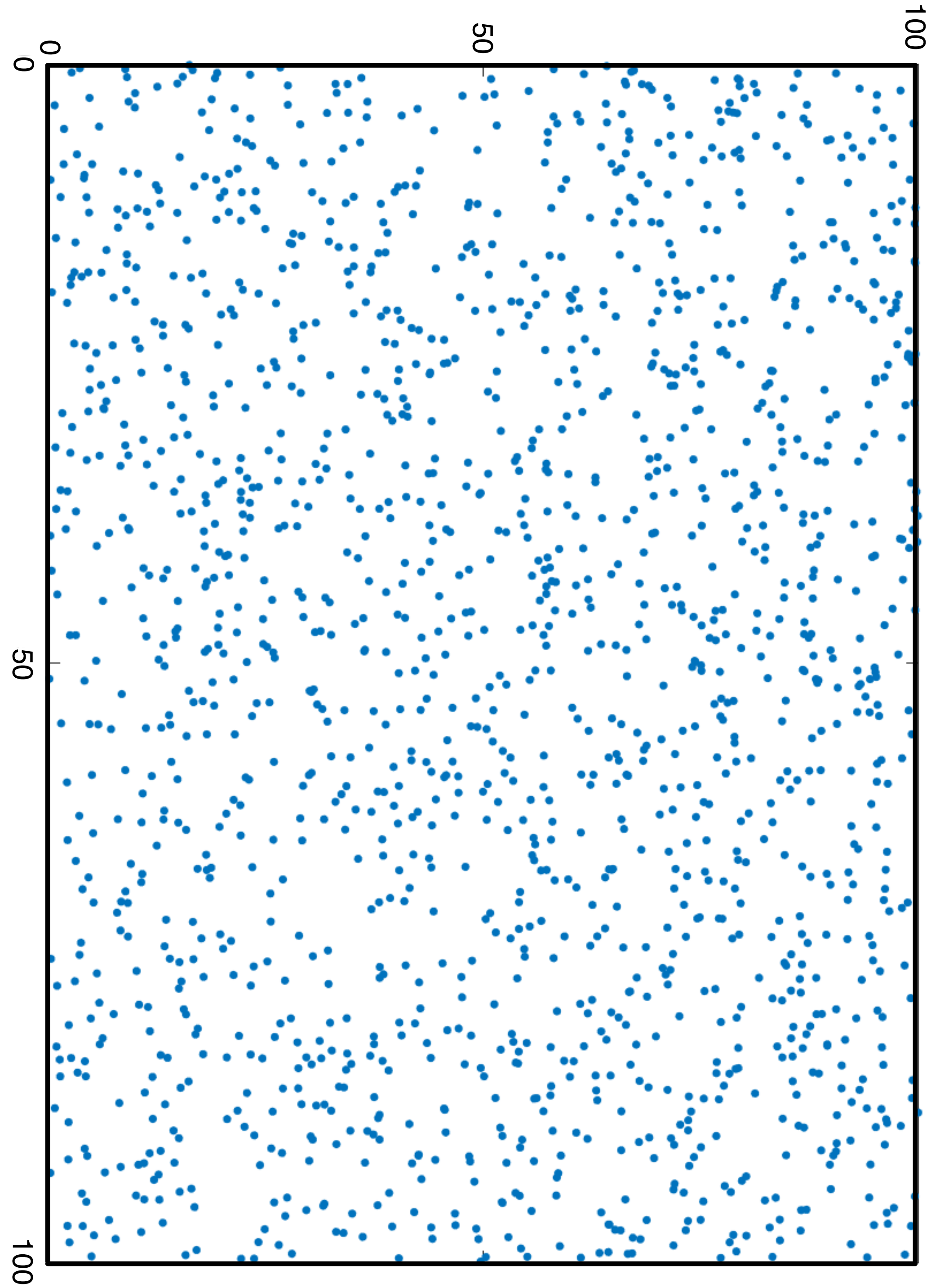

$\mathcal{P}_n$

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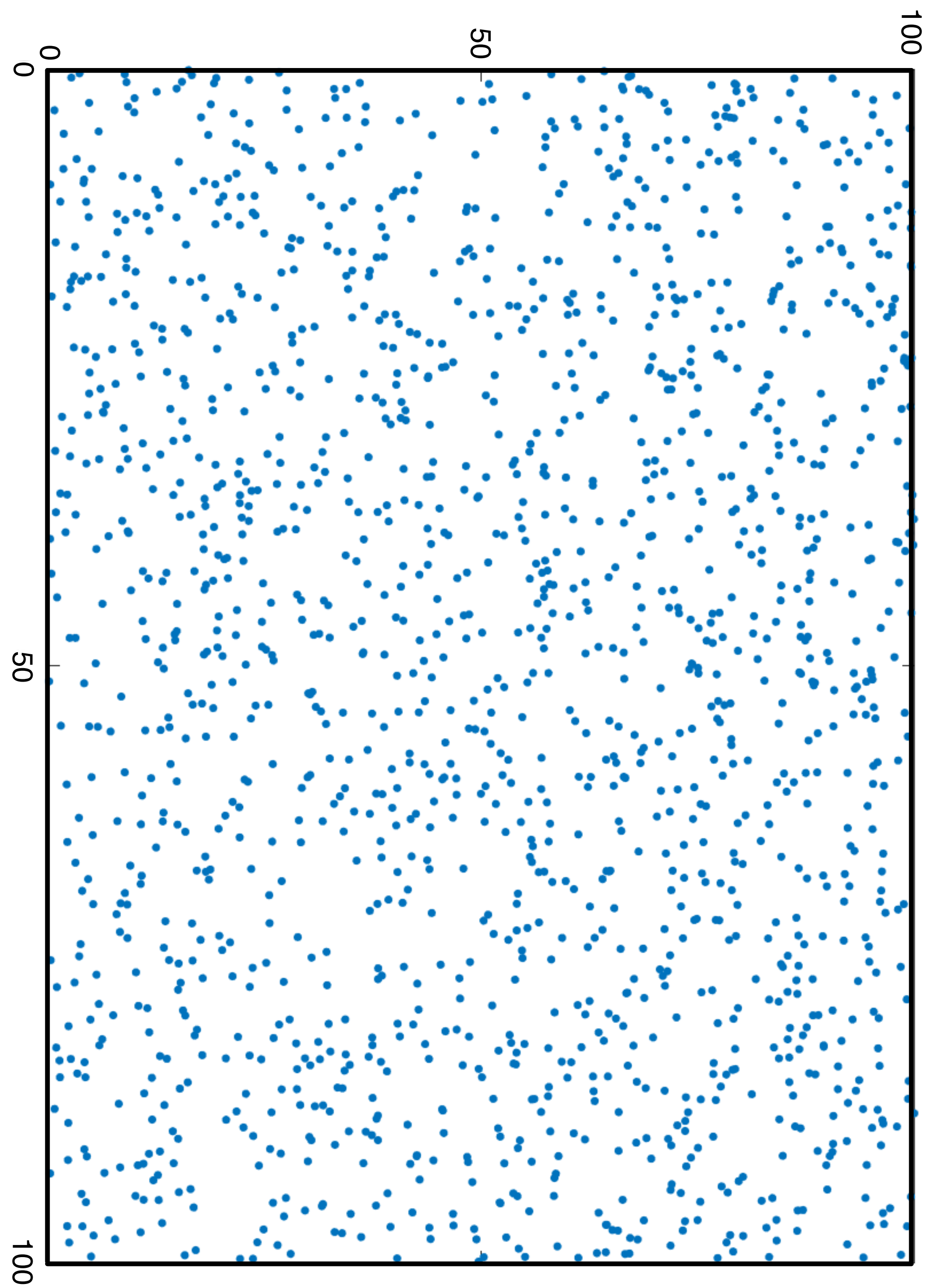

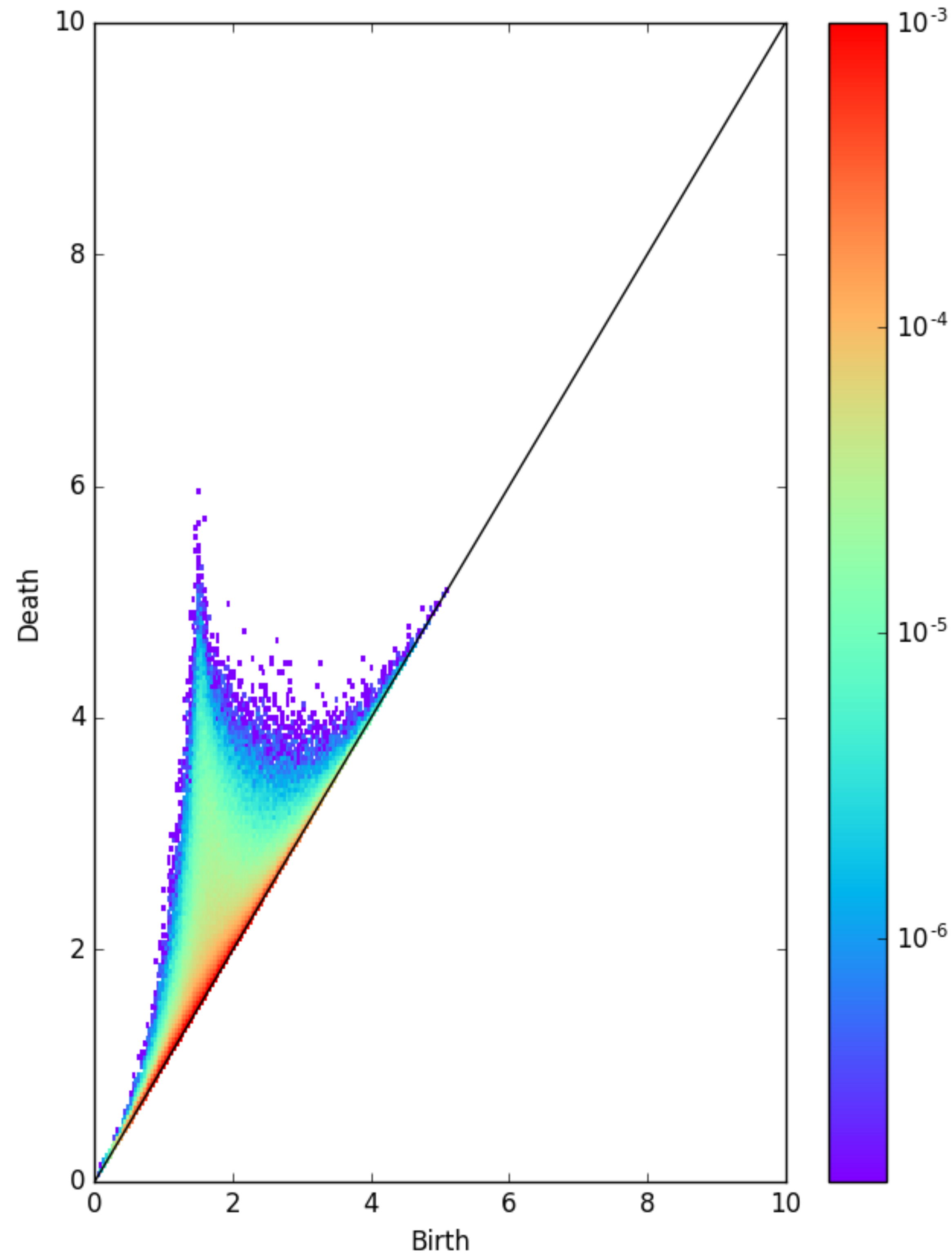

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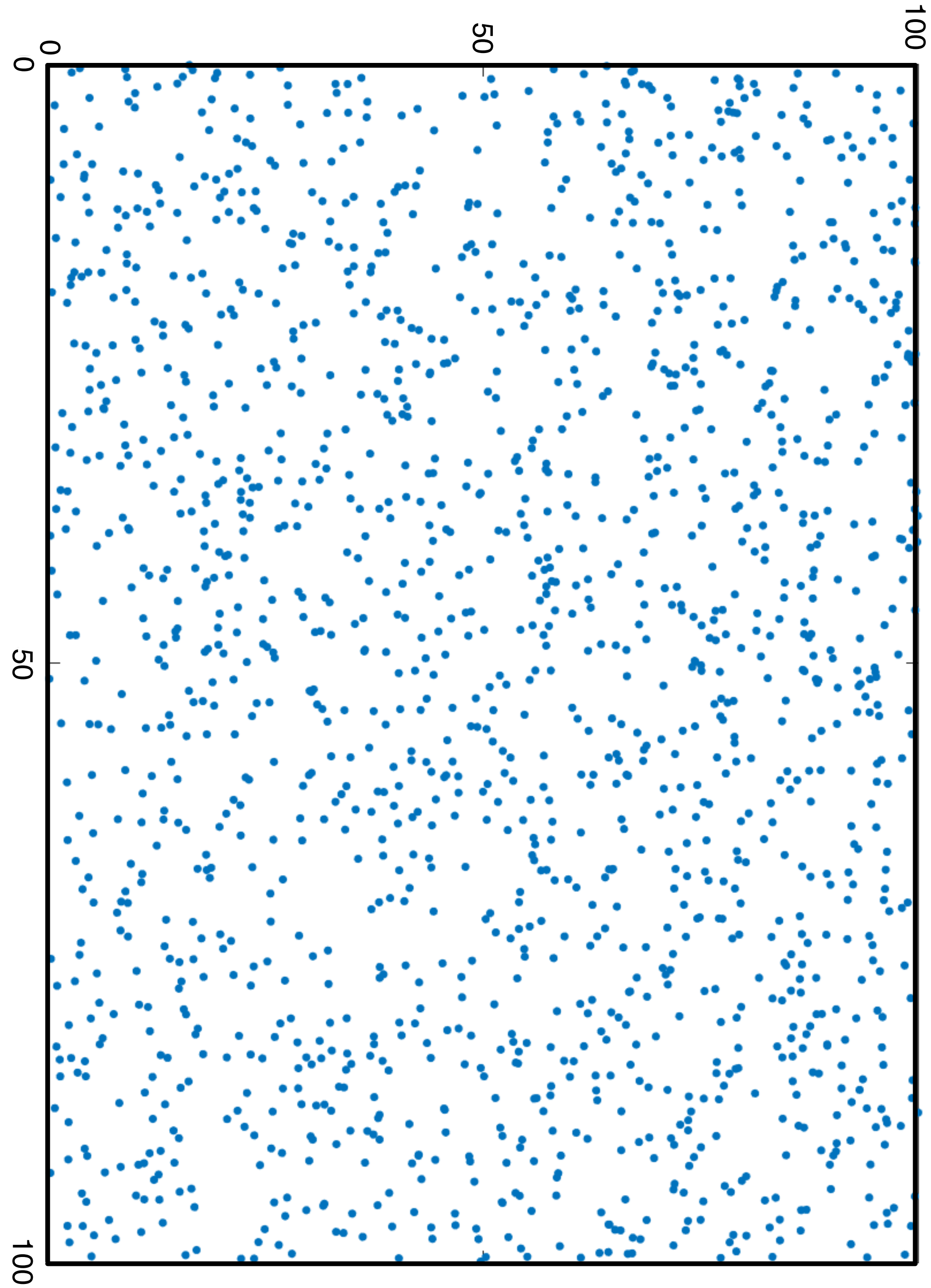

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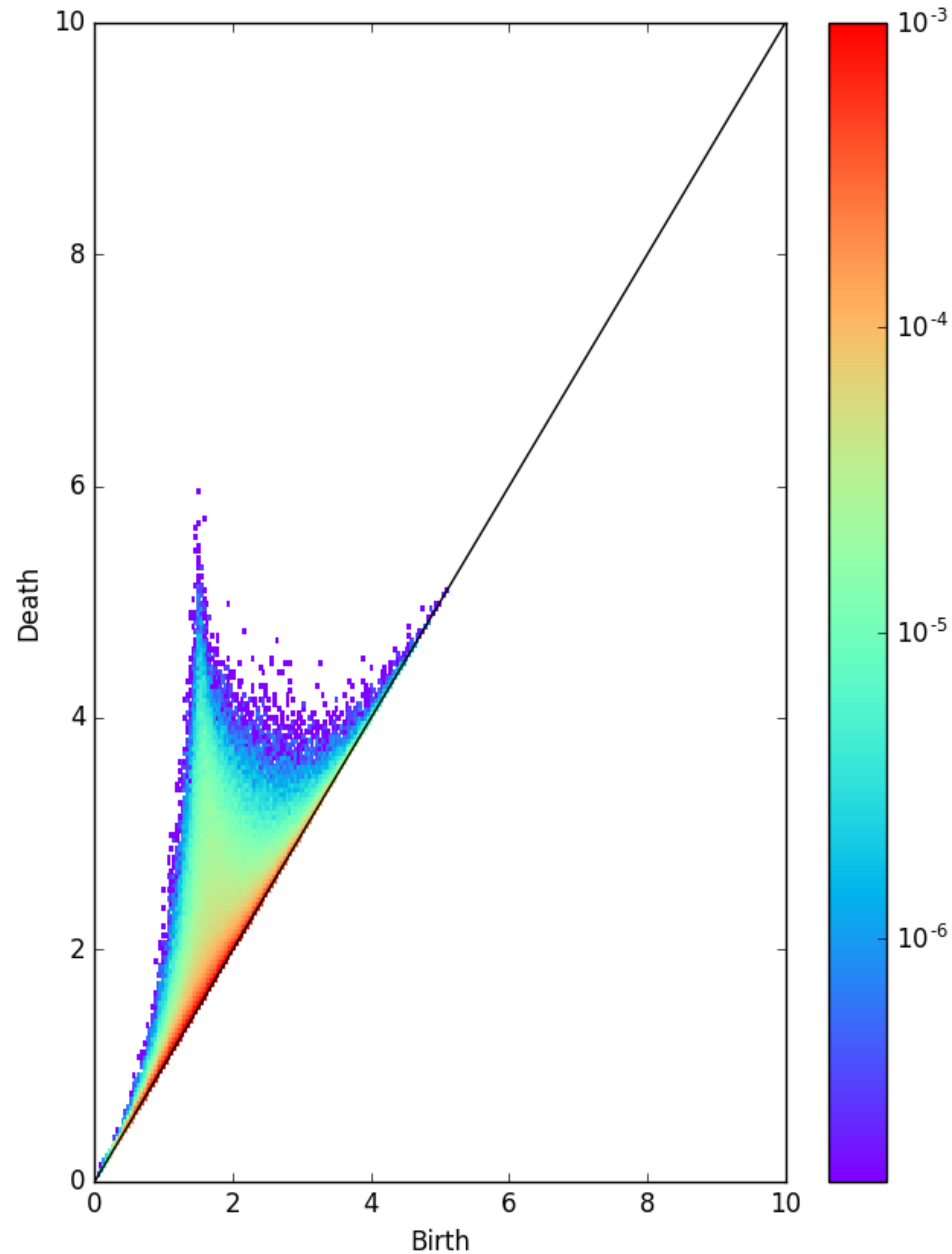

Density plot of $\frac{\mathcal{PD}_1(n^{-1/2}.)}{n}$

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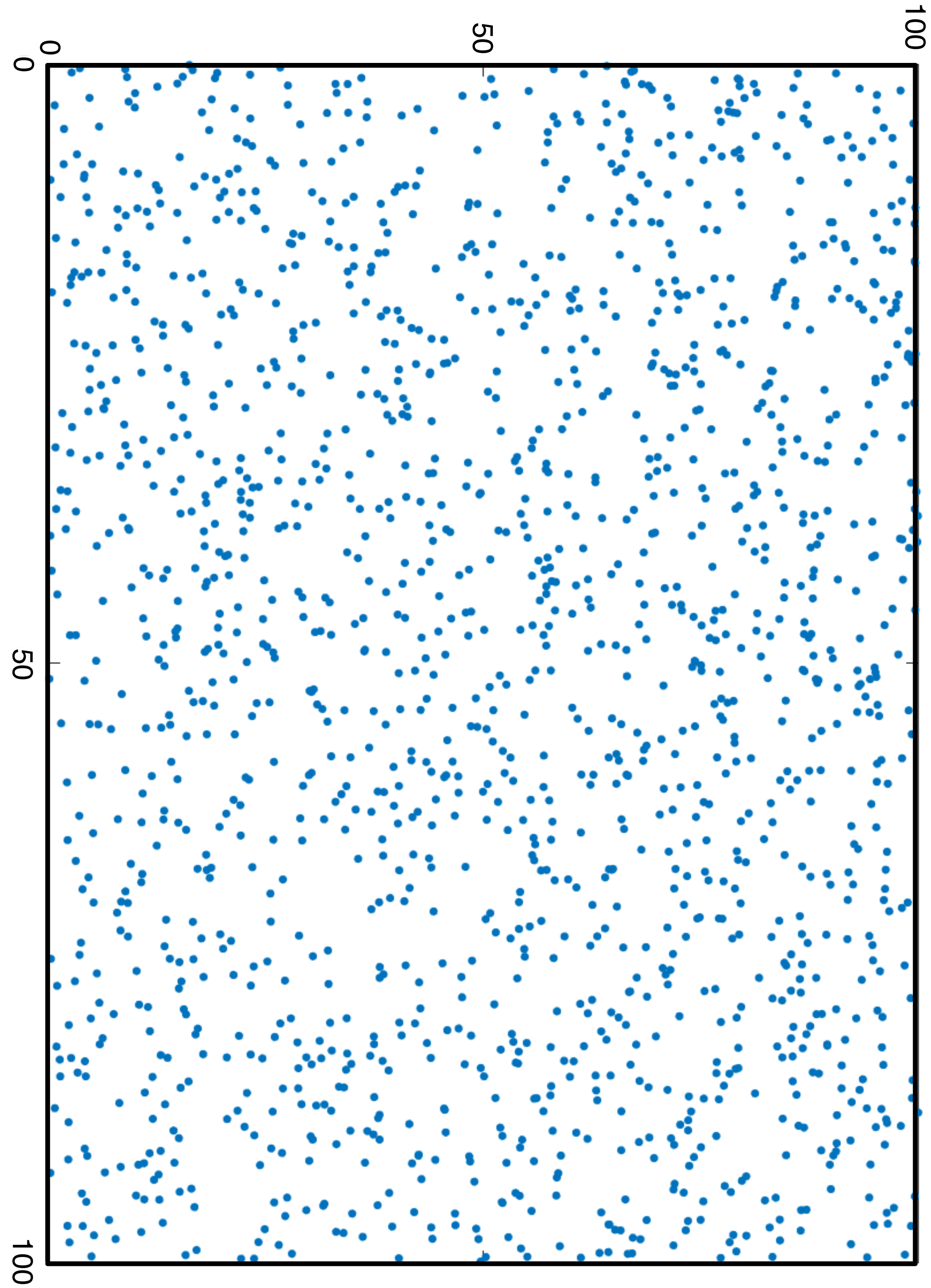

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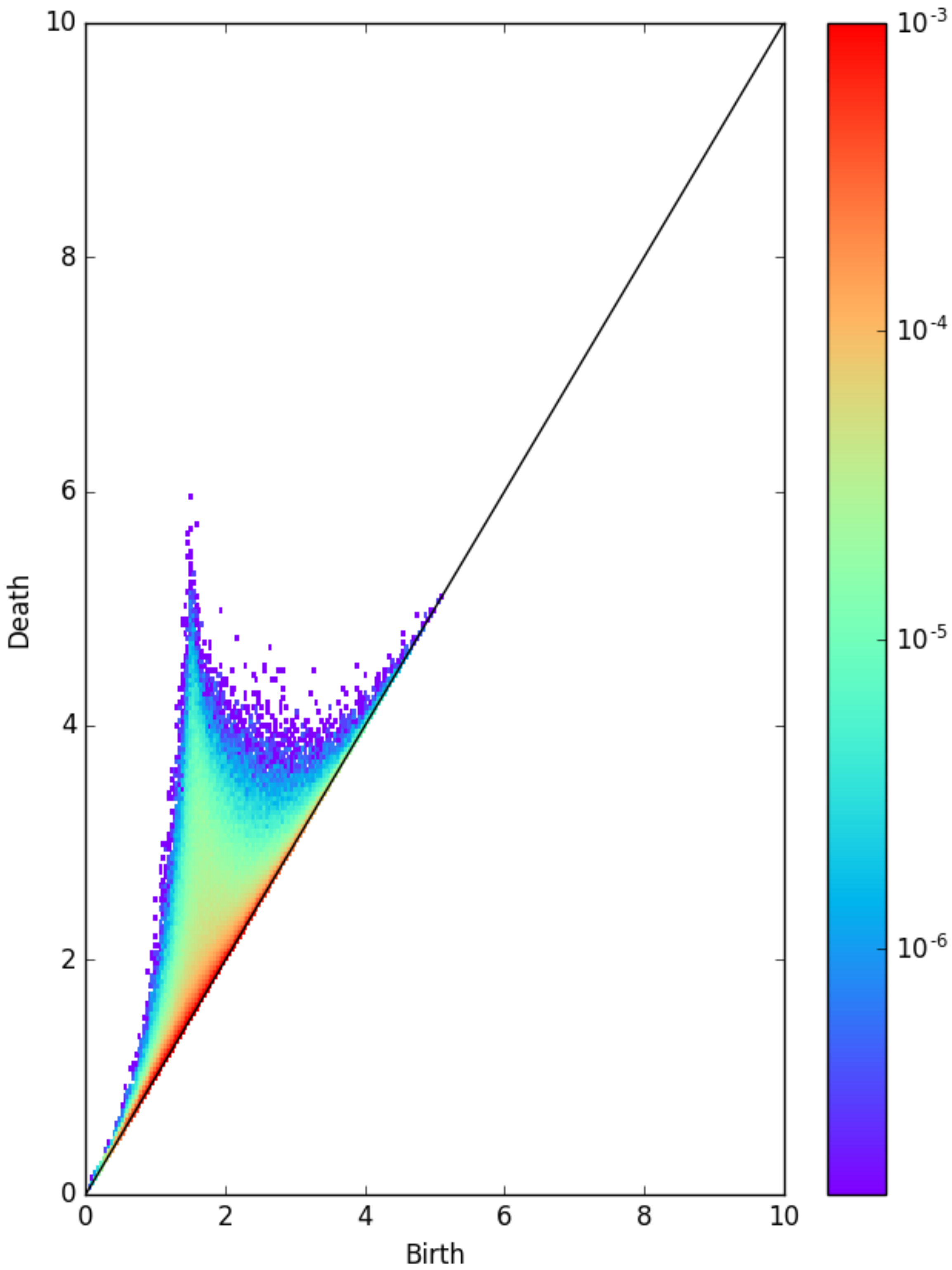

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Persistent Betti Number

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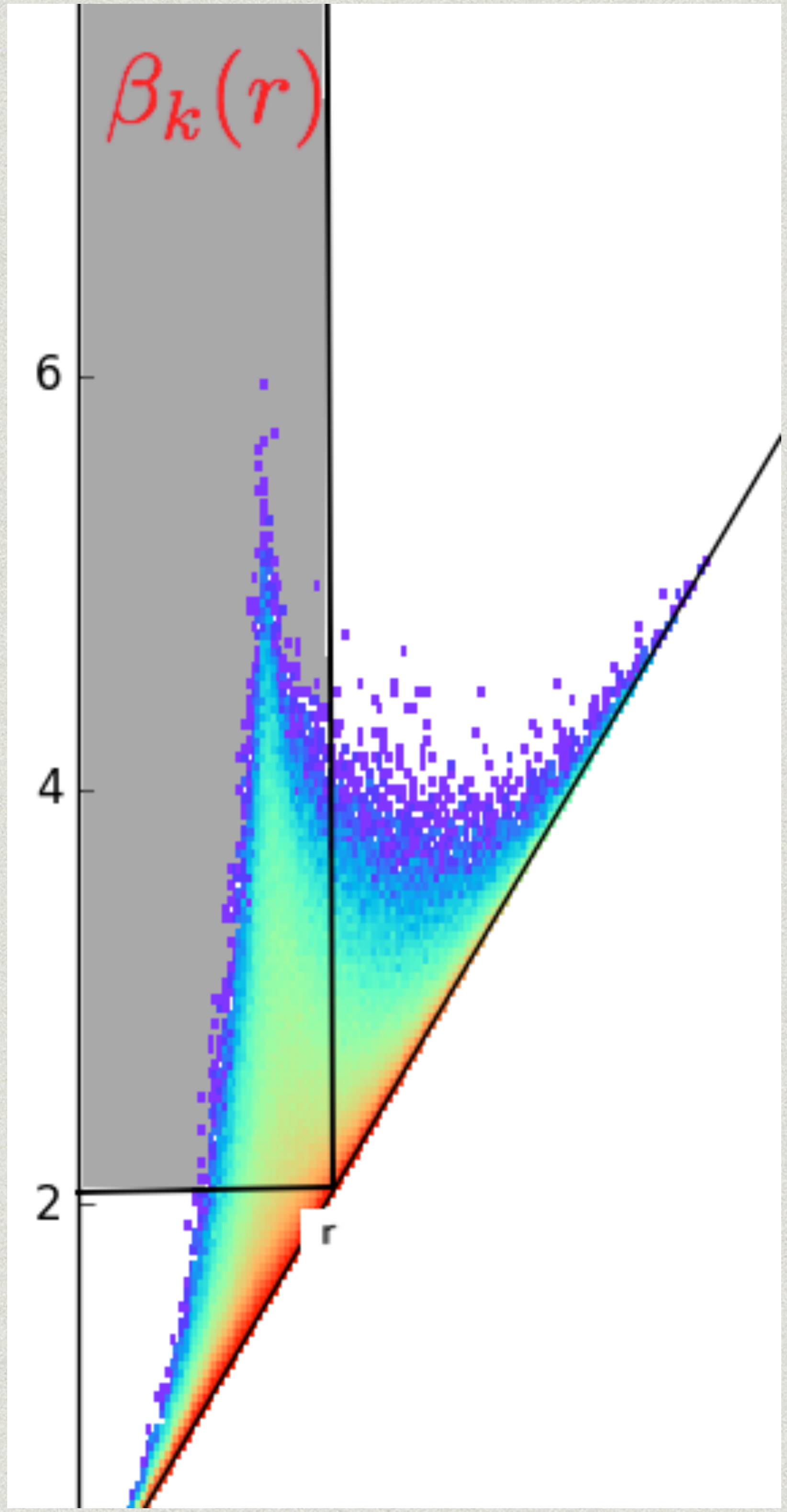

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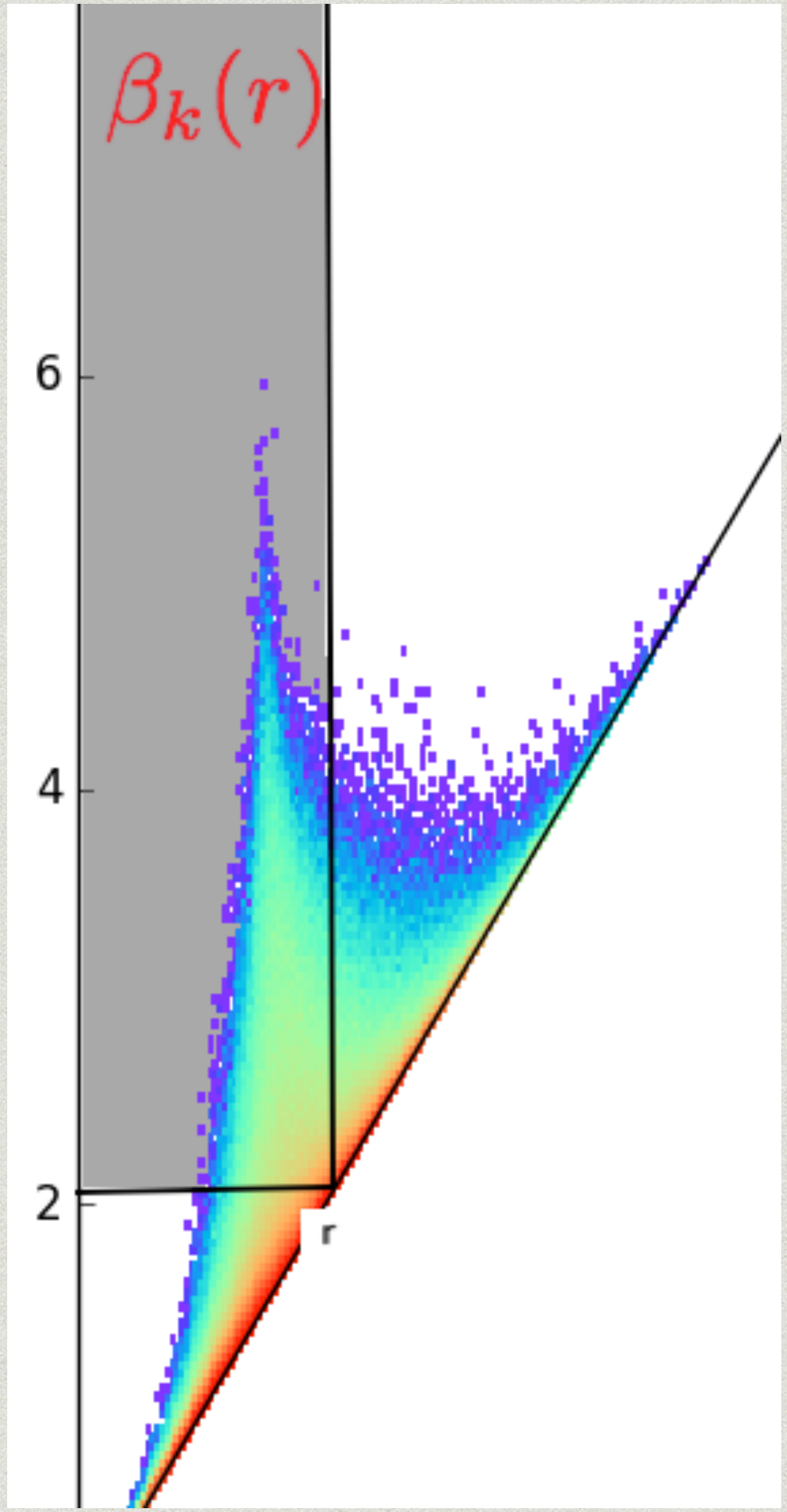

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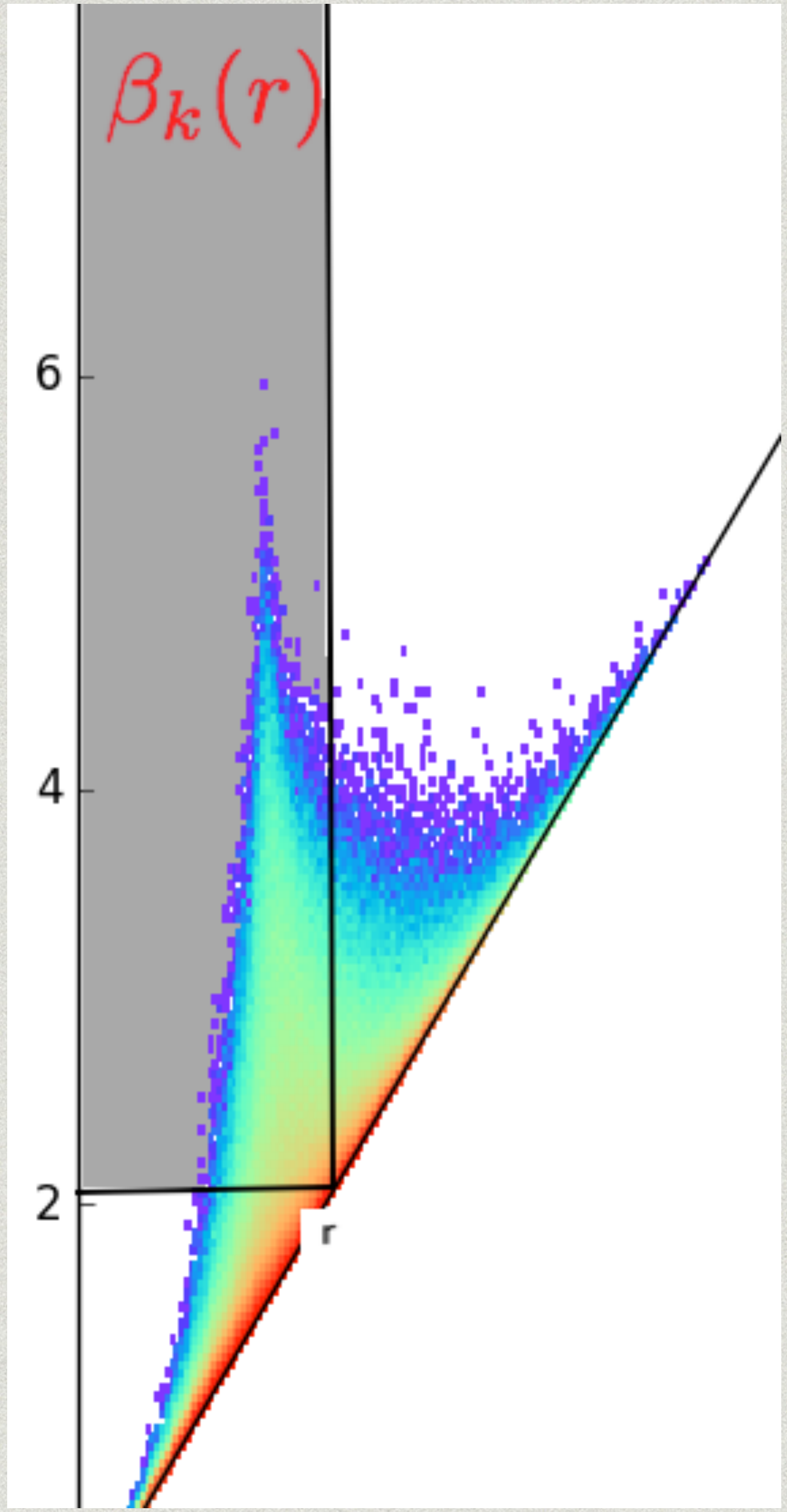

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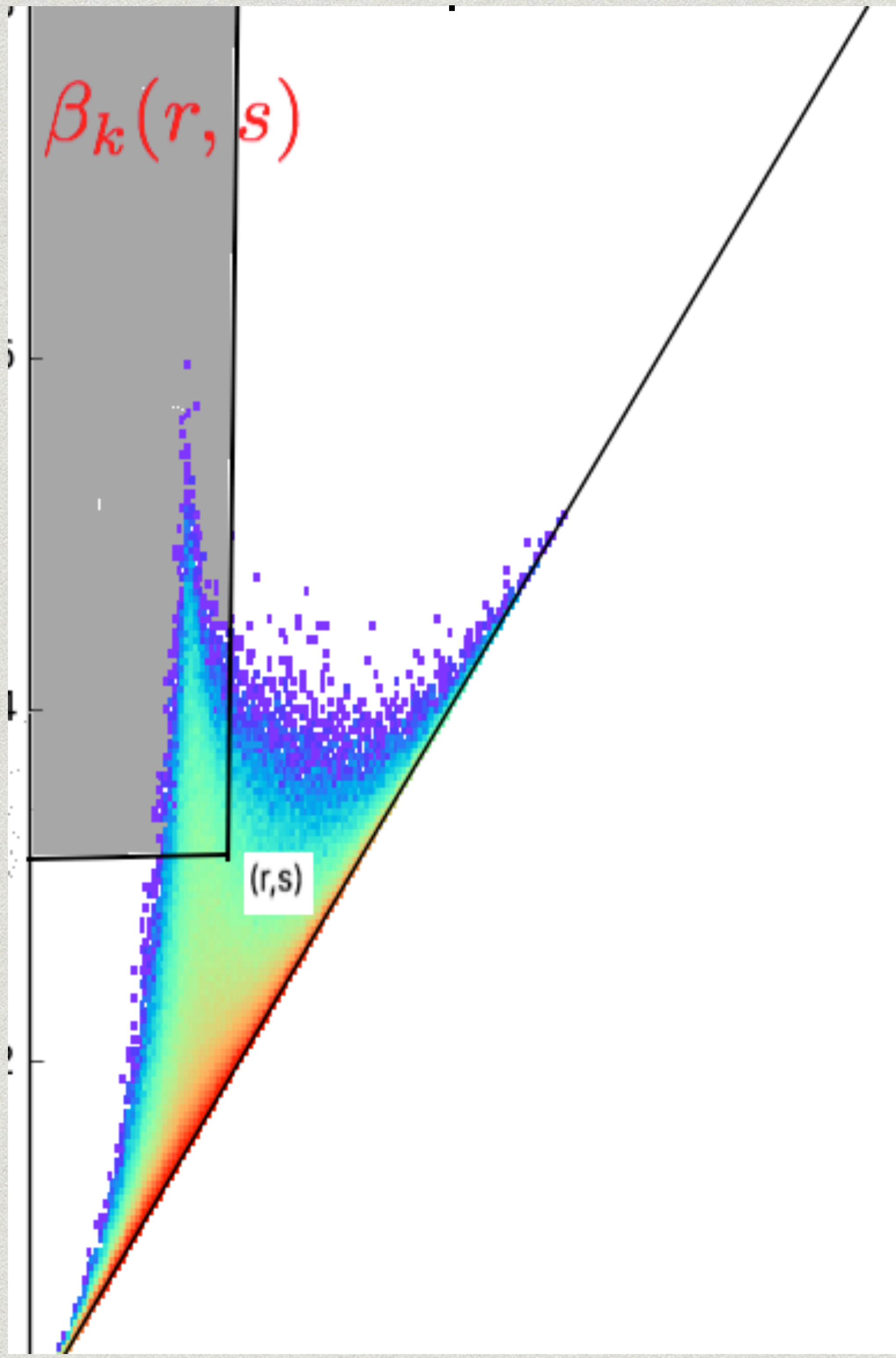
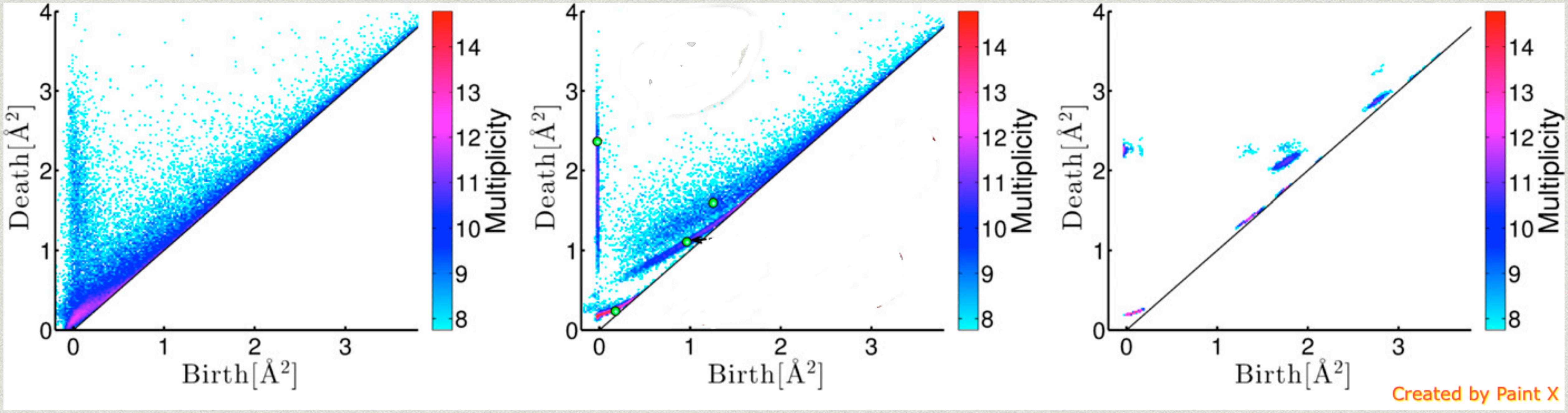
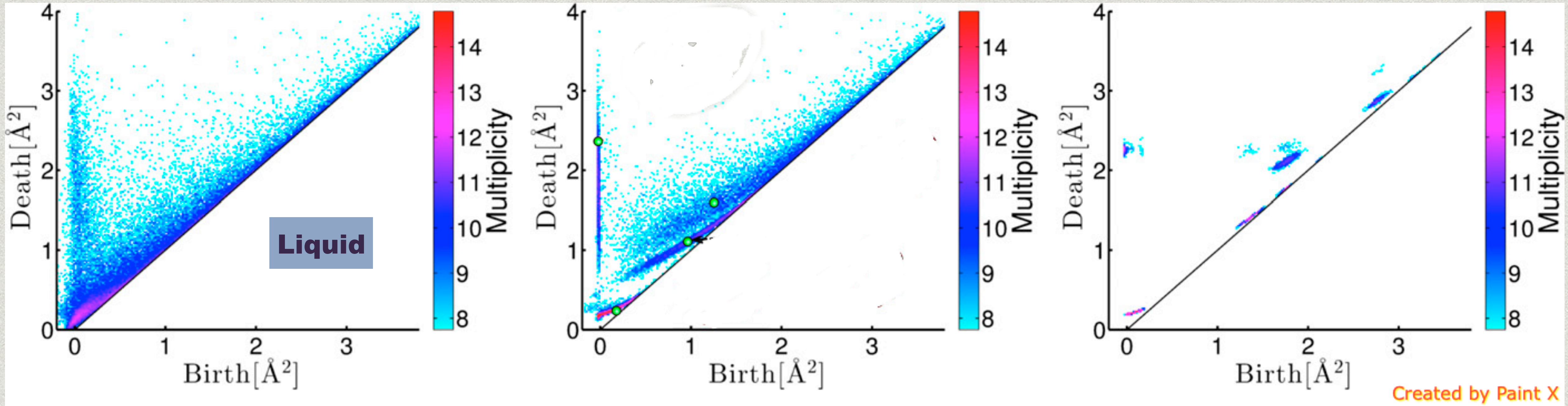




Death[Å²]
Birth[Å²]
Multiplicity
Created by Paint X



Liquid
Death[Å²]
Birth[Å²]
Multiplicity
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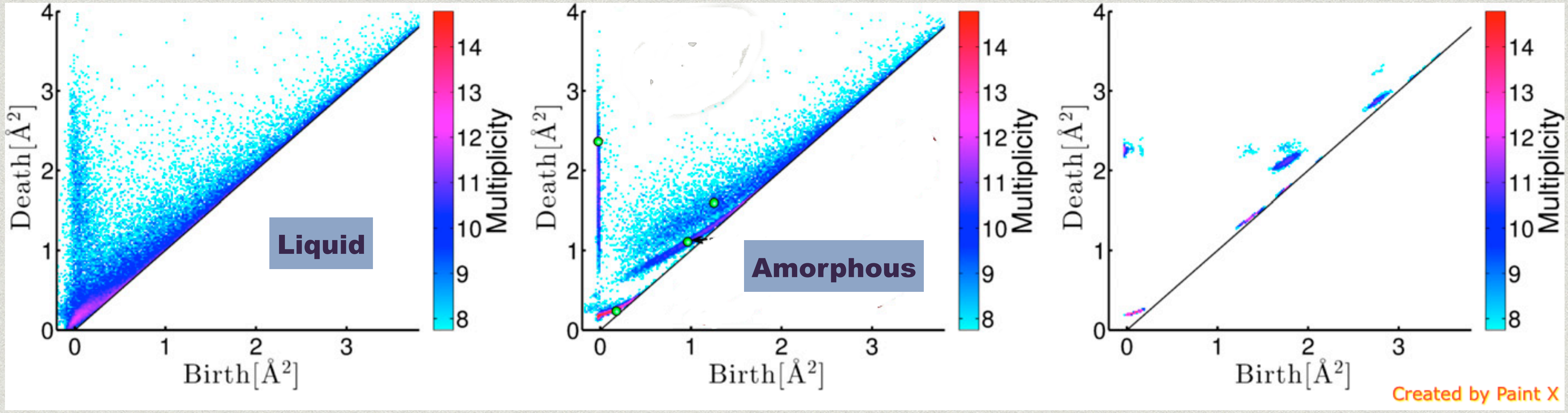

Liquid
Amorphous
Death[Å^2]
Birth[Å^2]
Multiplicity
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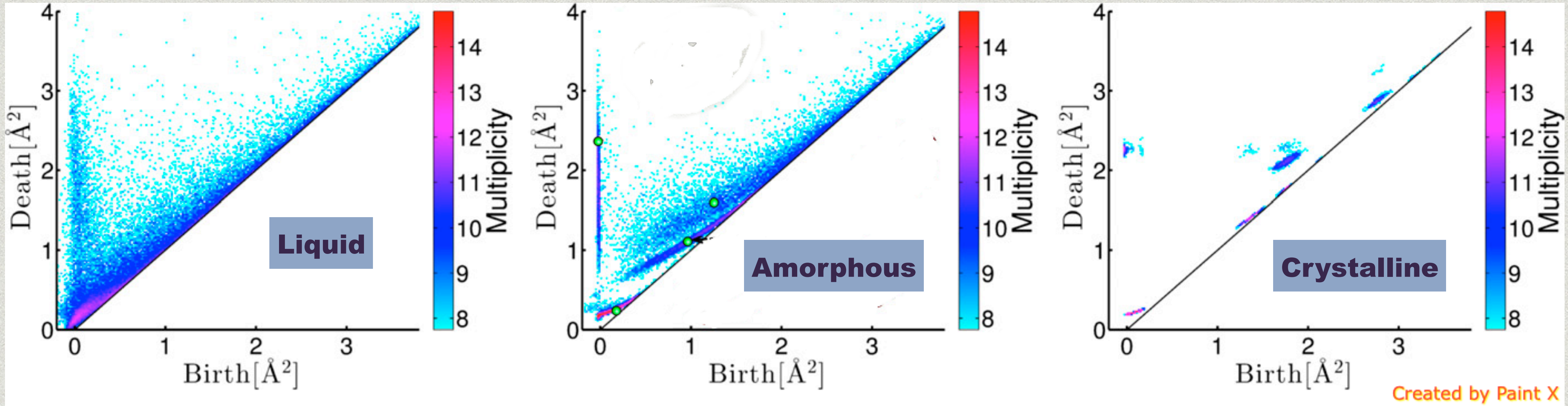
Liquid
Amorphous
Crystalline
Death[Å²]
Birth[Å²]
Multiplicity
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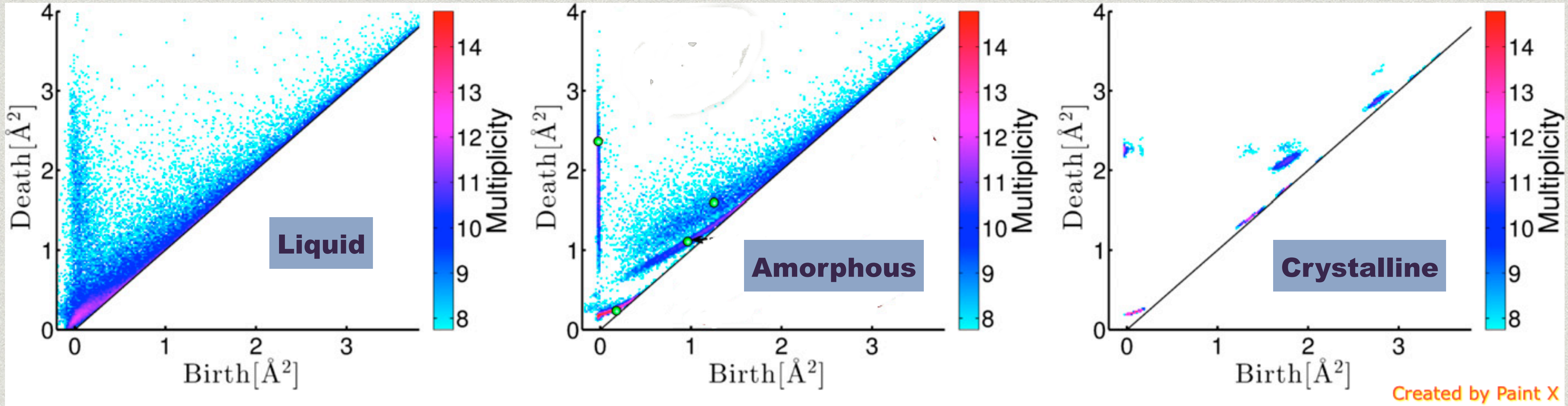
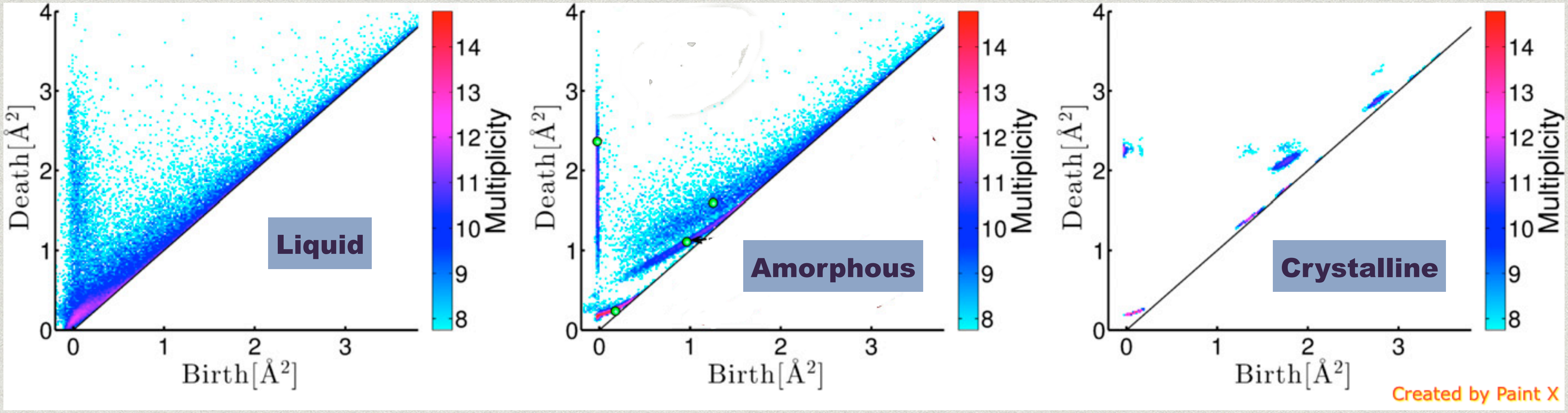


Hiraoka et al : Hierarchical structure of amorphous solids…..



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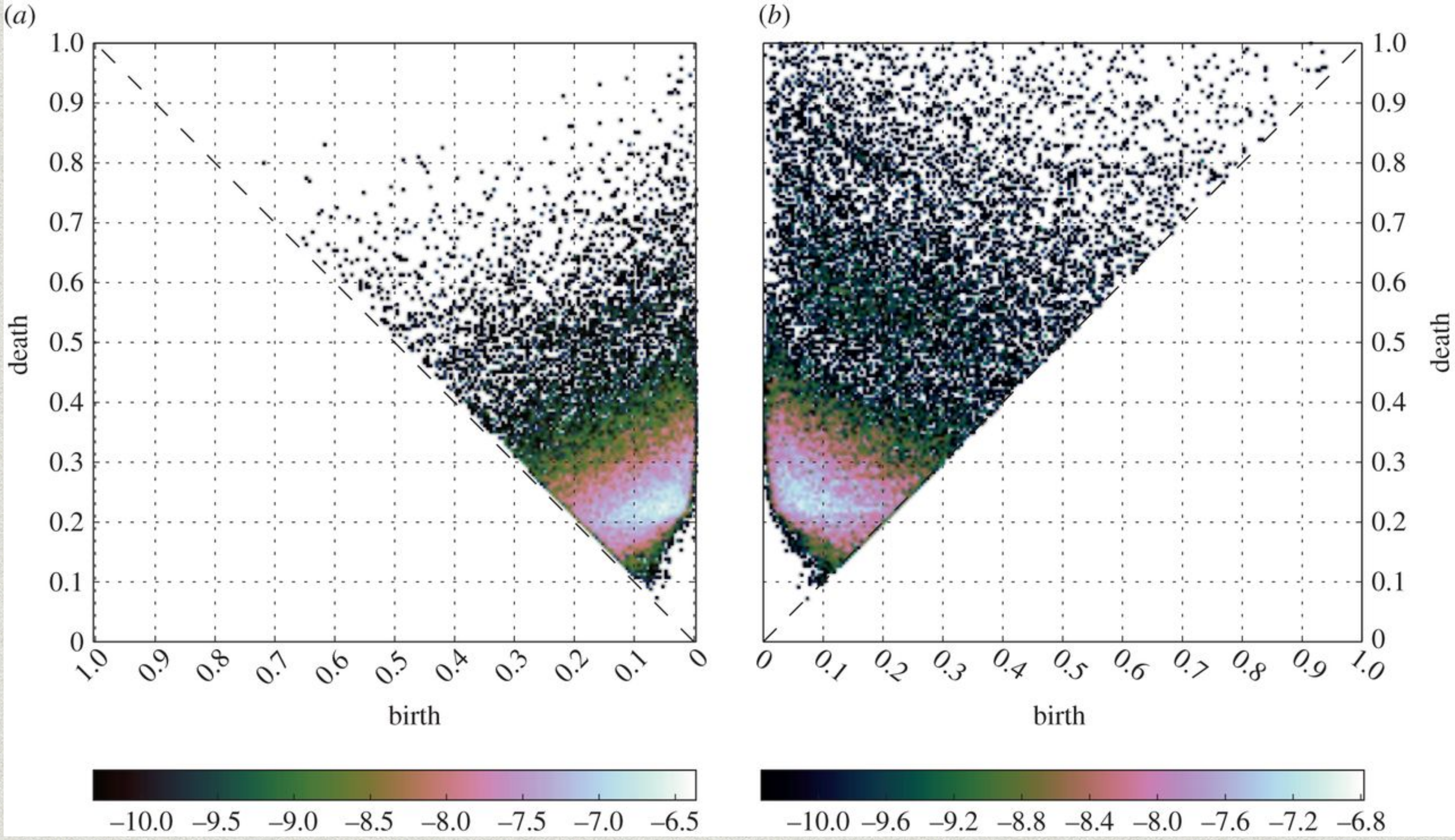

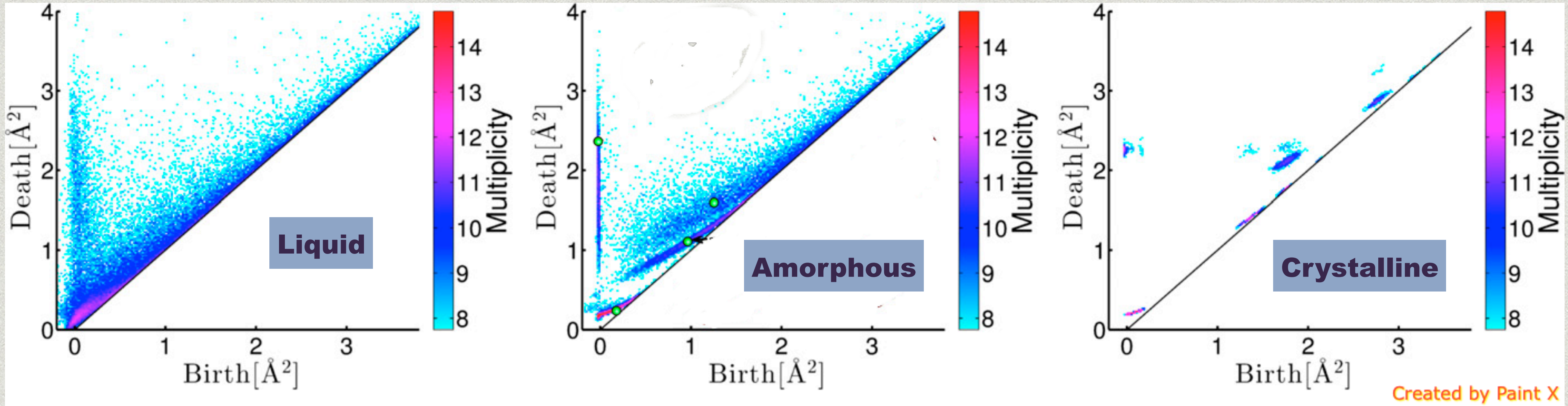

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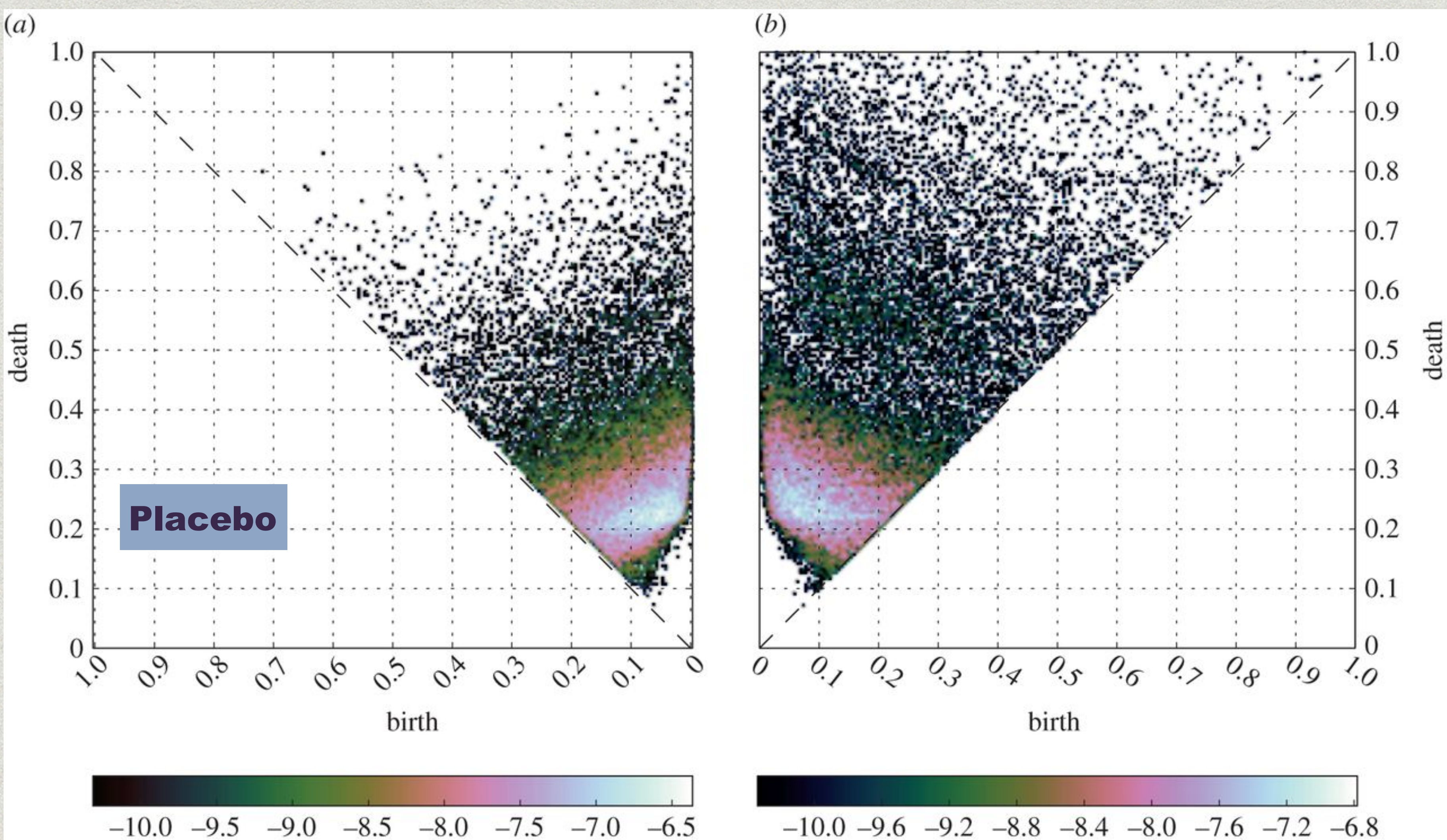

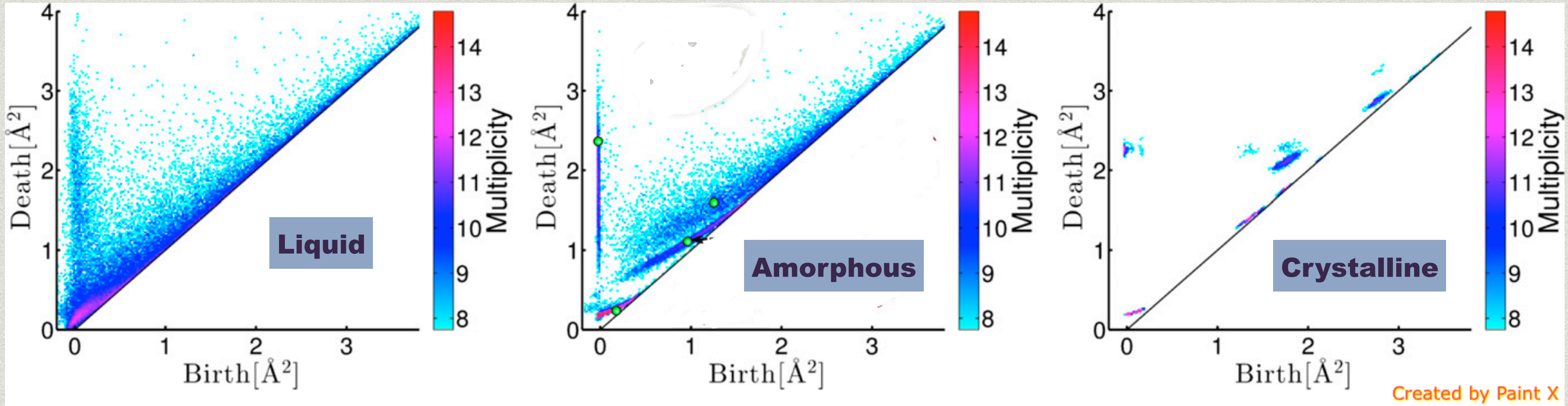

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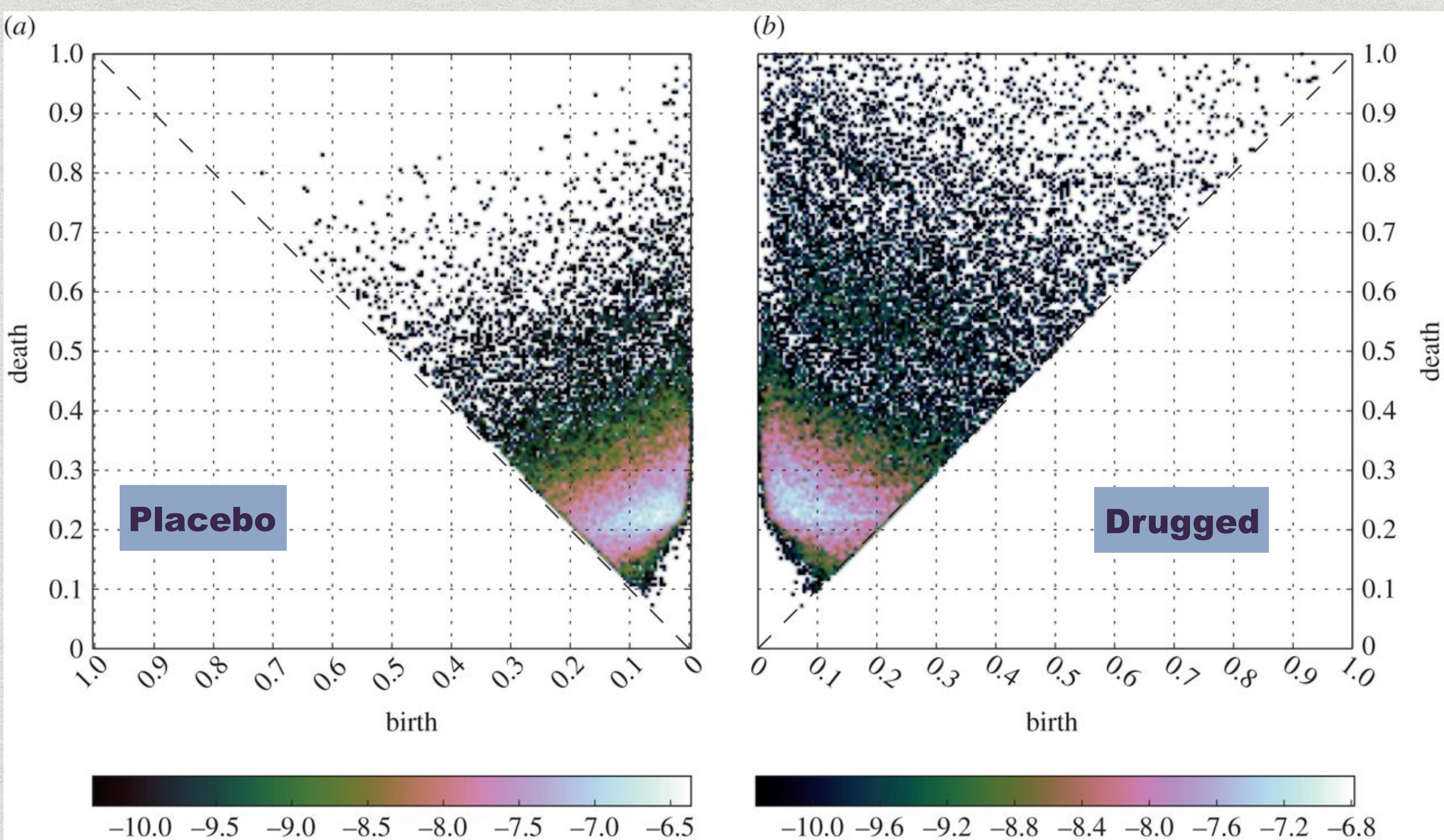

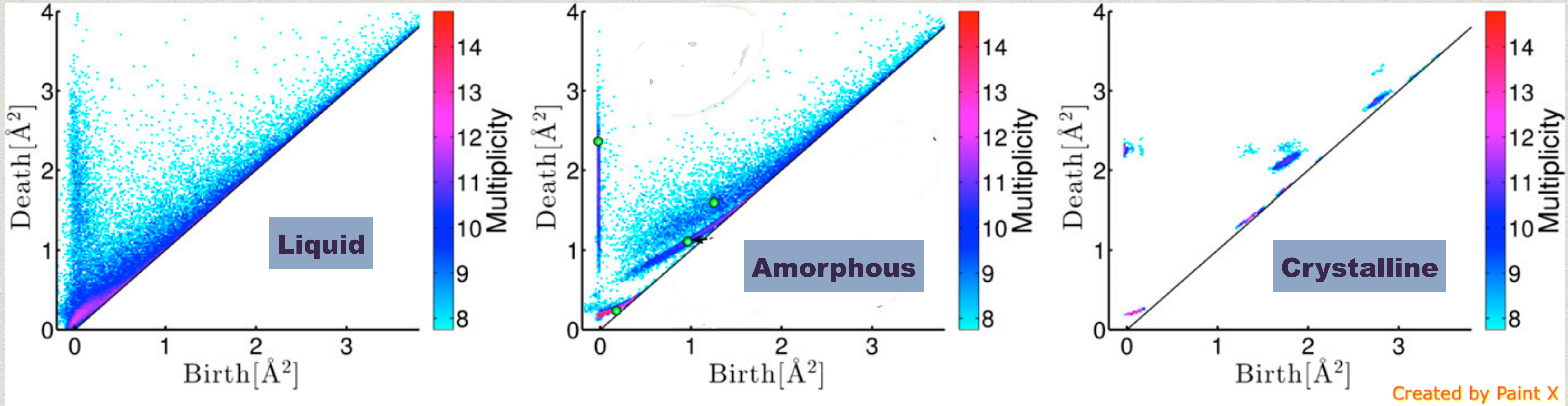

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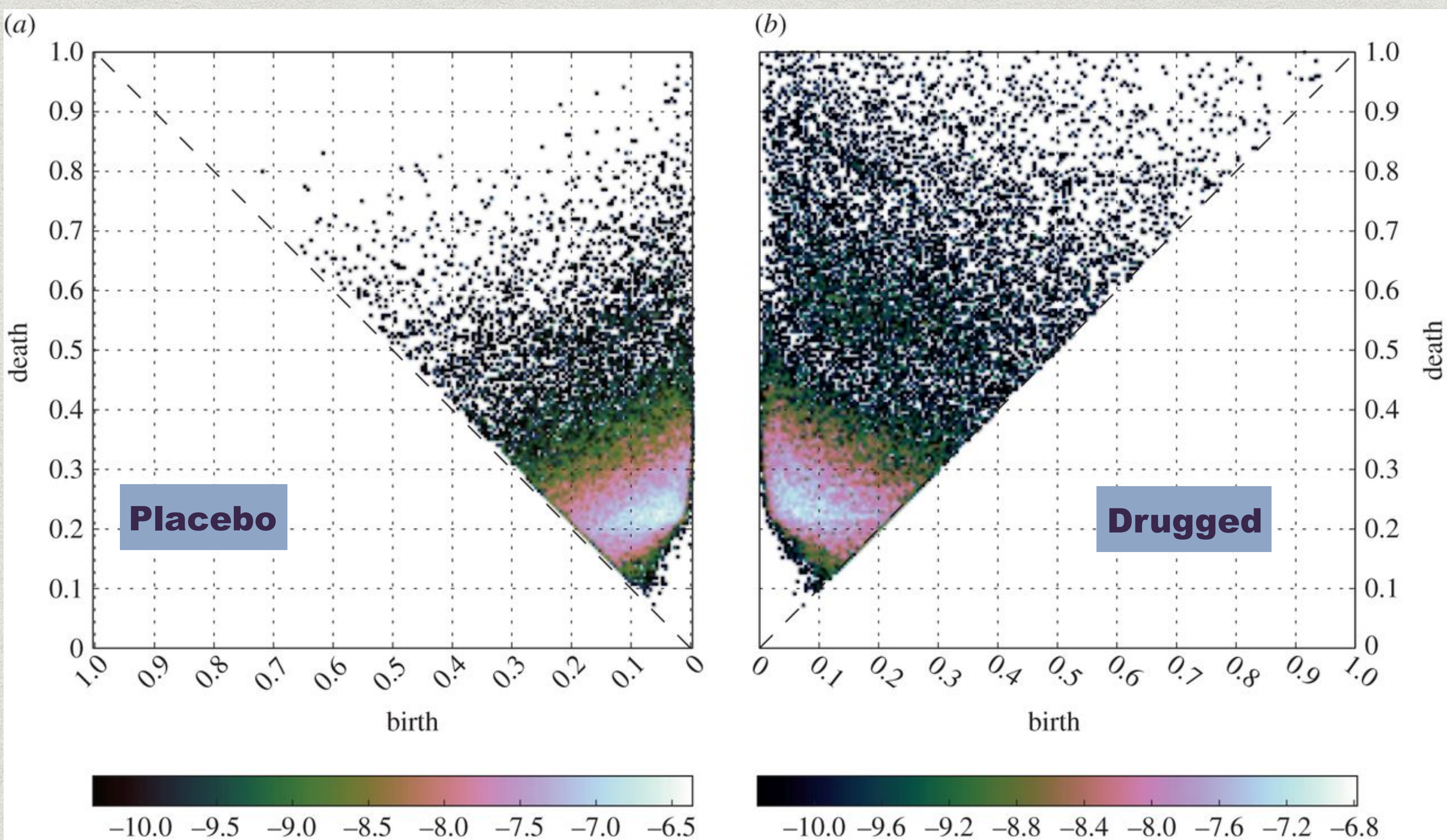
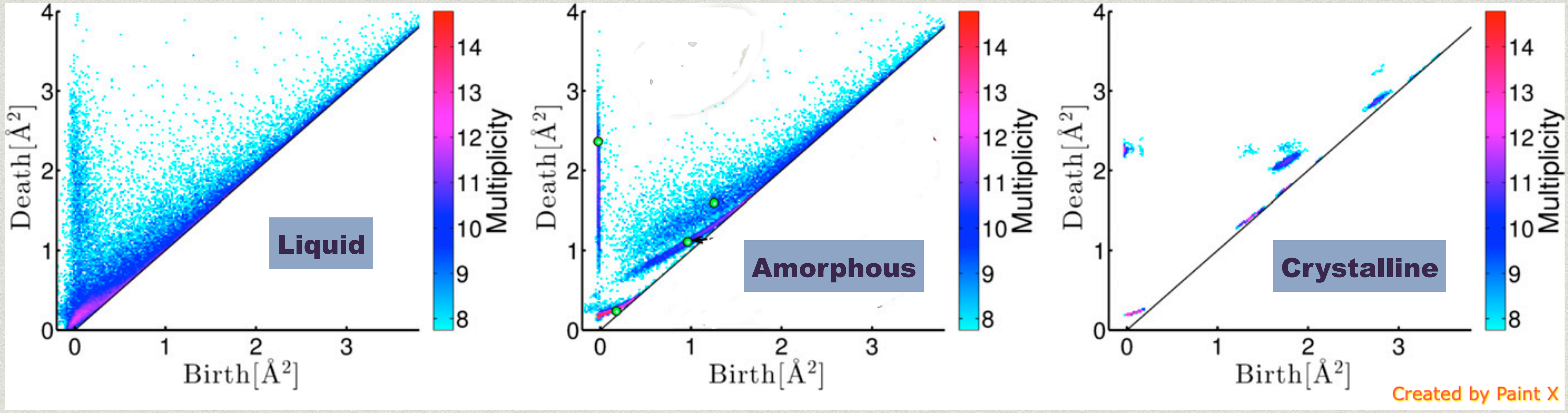


G. Petri et al.



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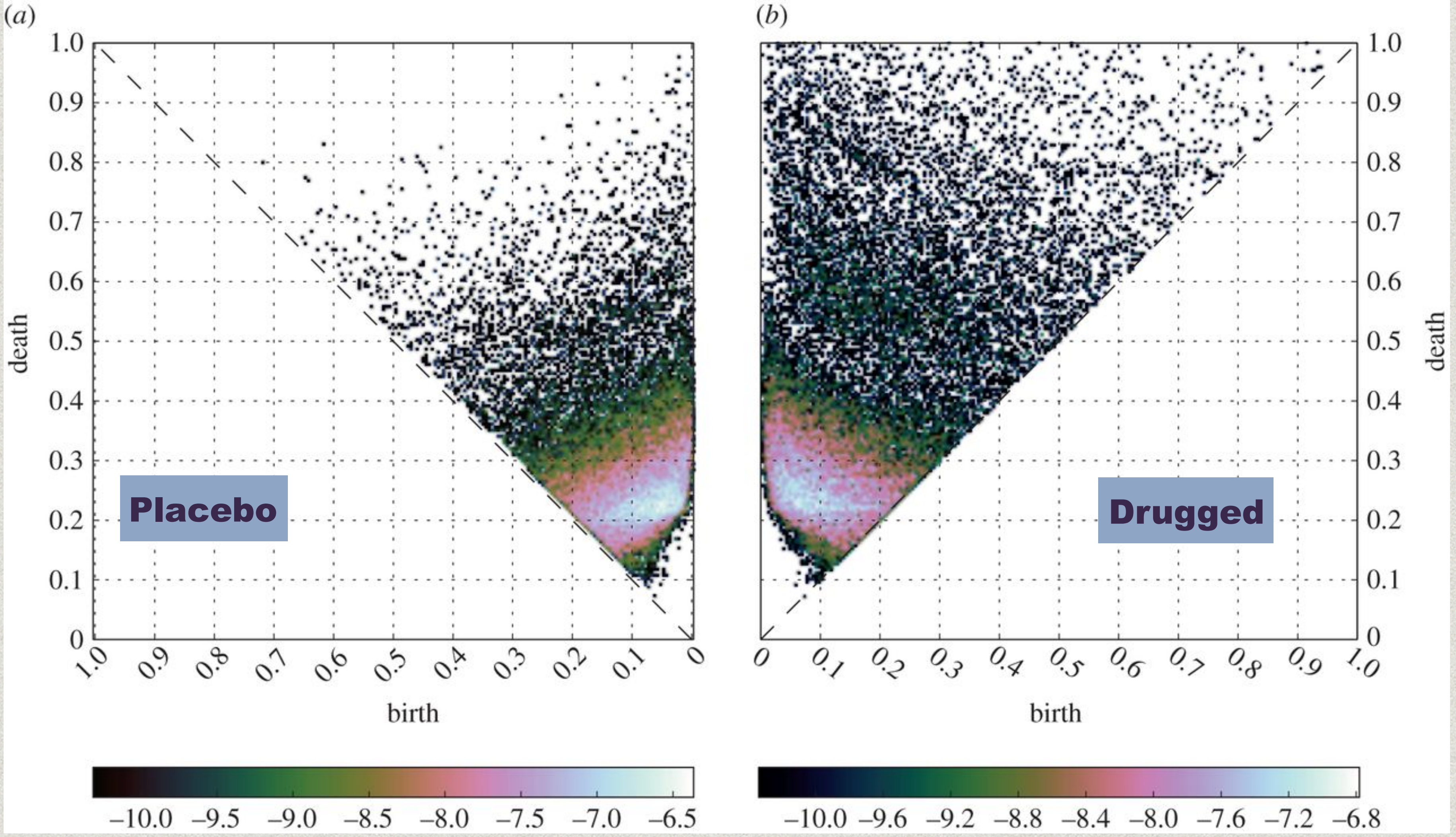

G. Petri et al.

Homological scaffolds for brain functional networks.

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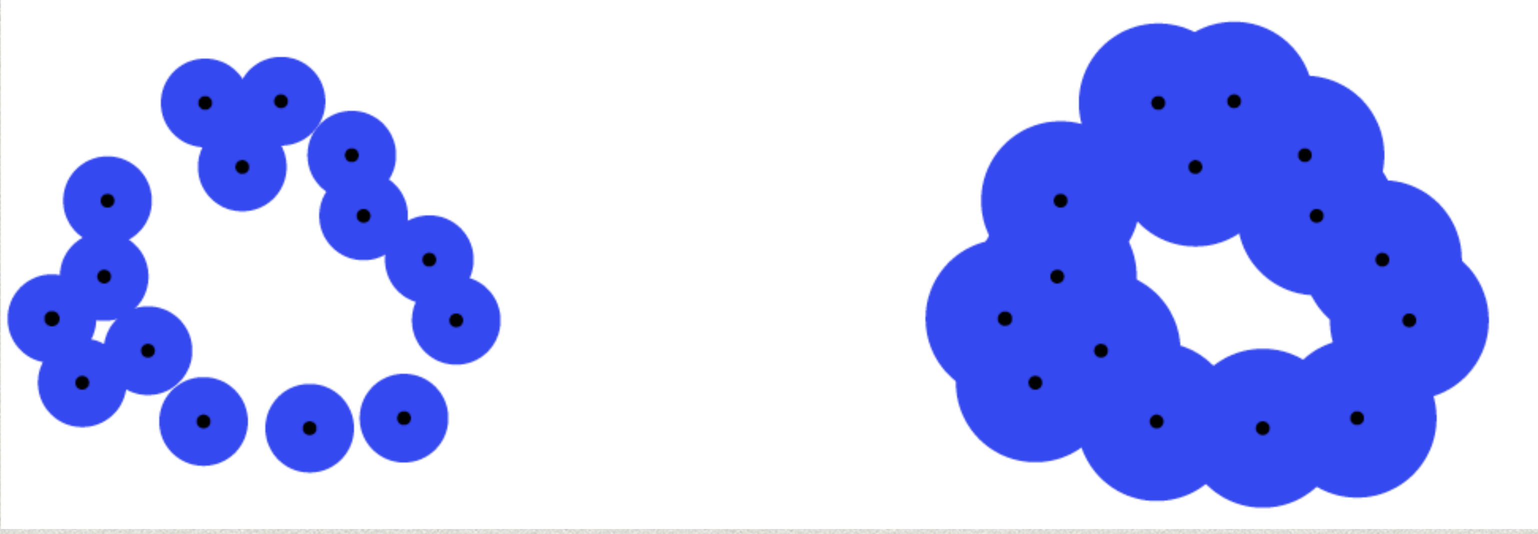

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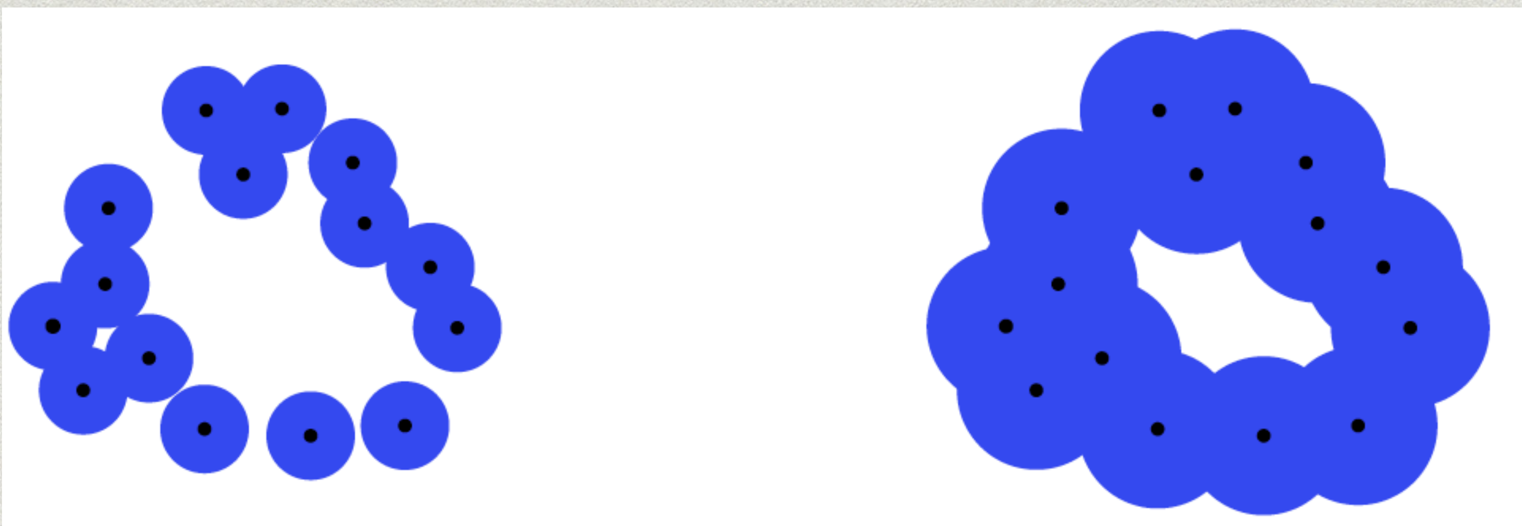

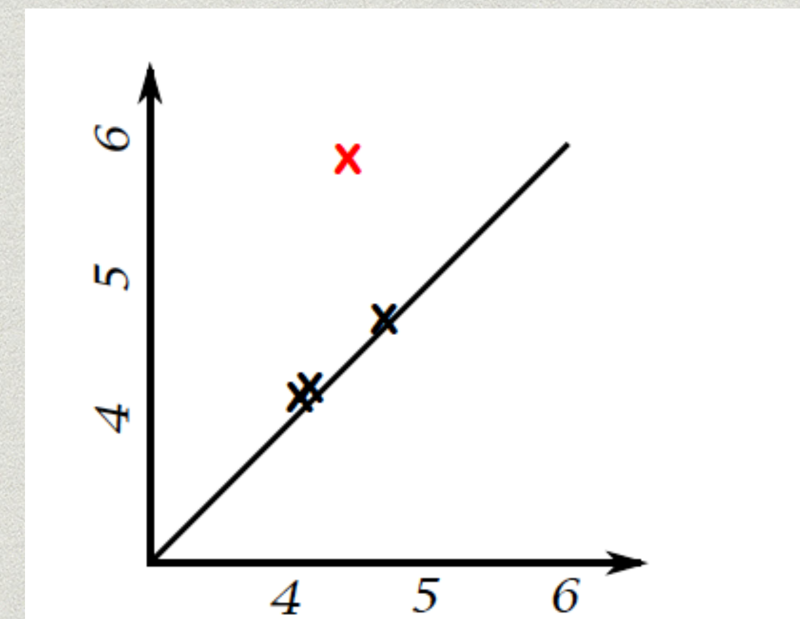

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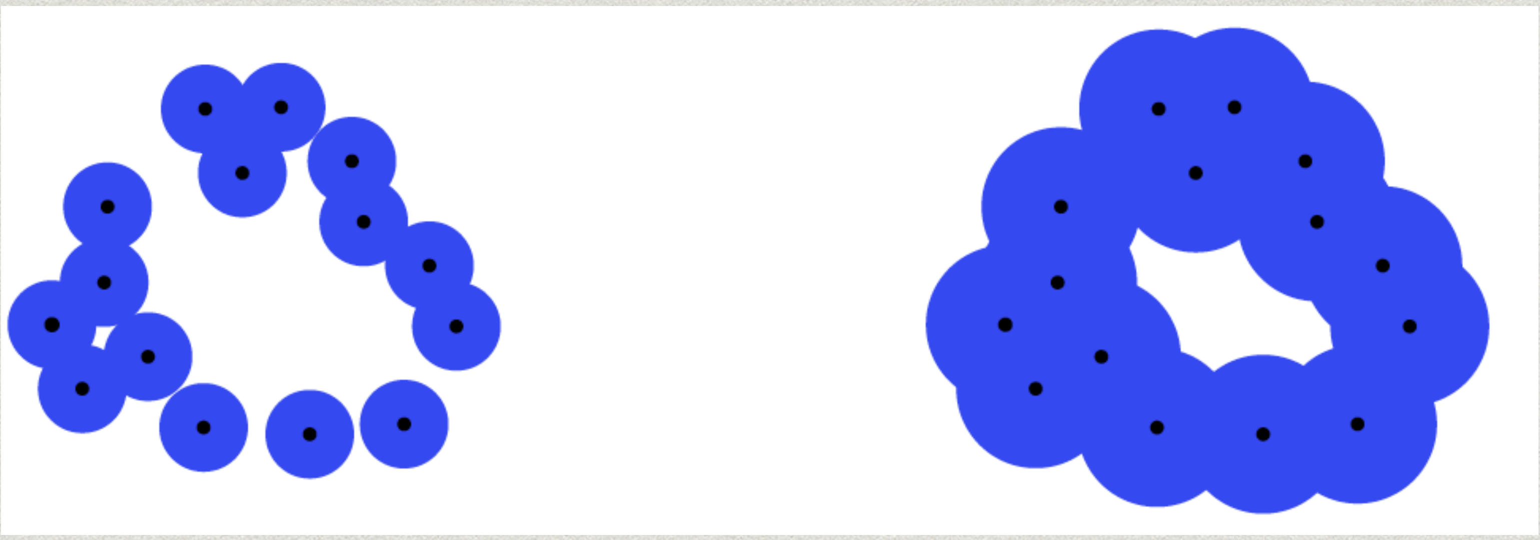

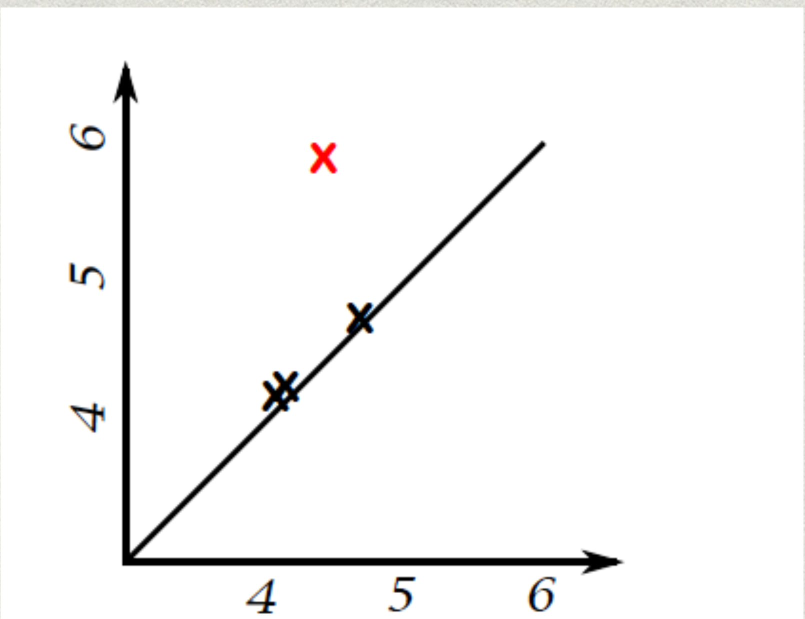

Paradigm : Long lifetimes —> Signal / Information ; Short lifetimes —> Noise

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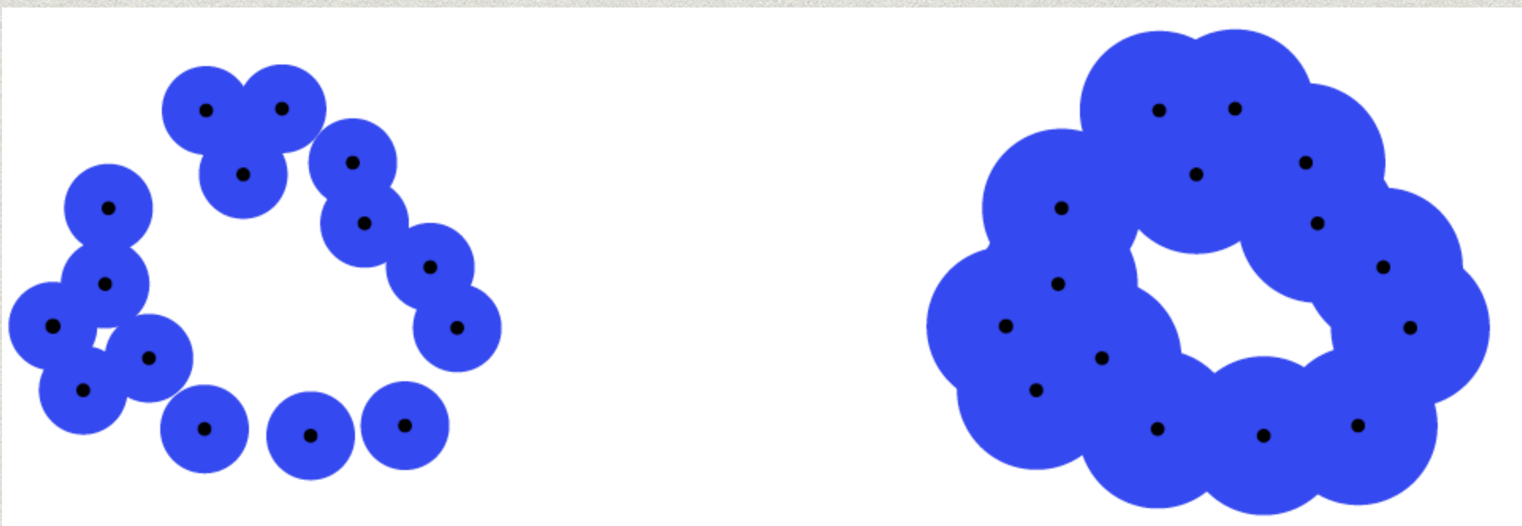

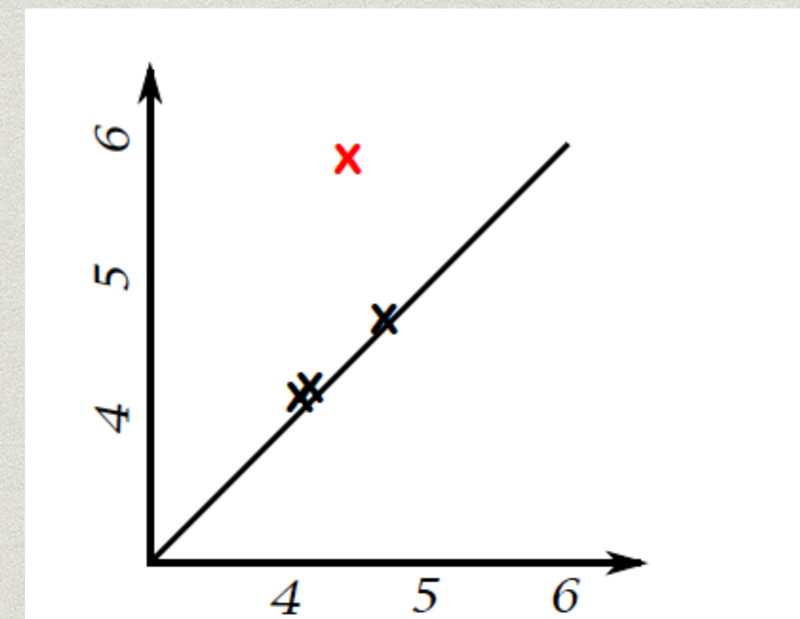

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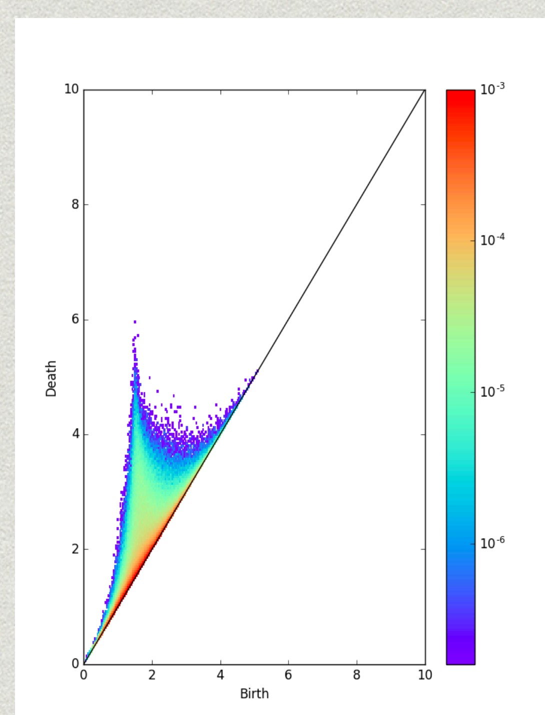

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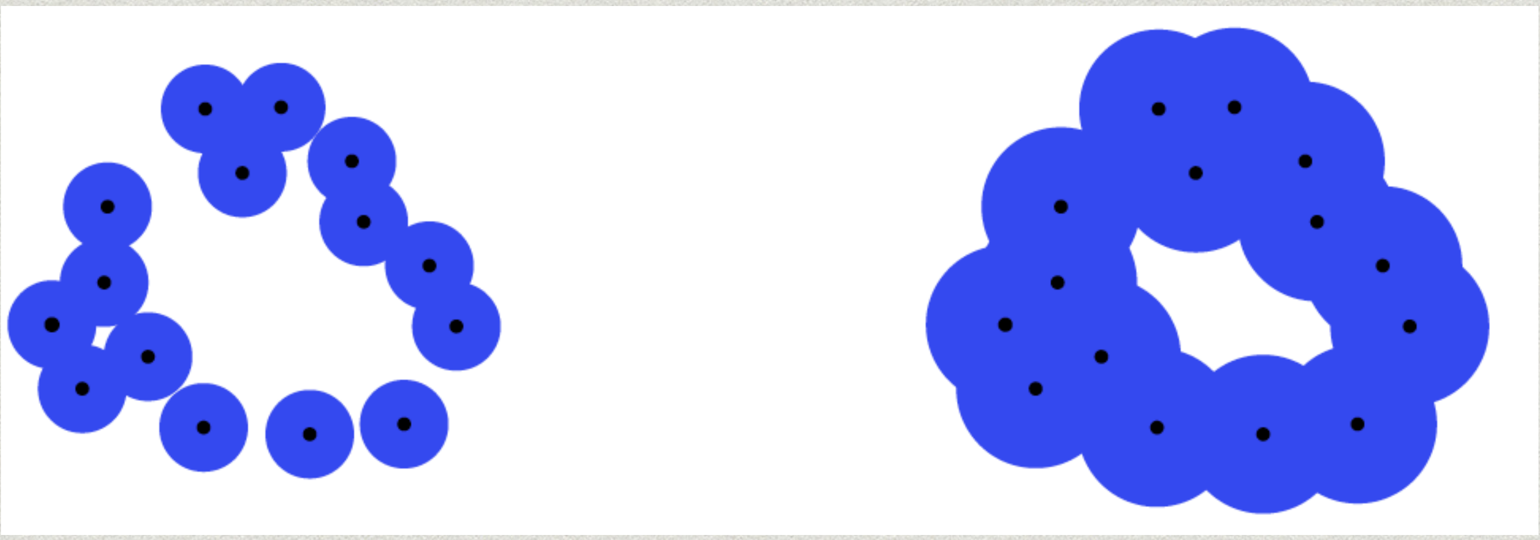

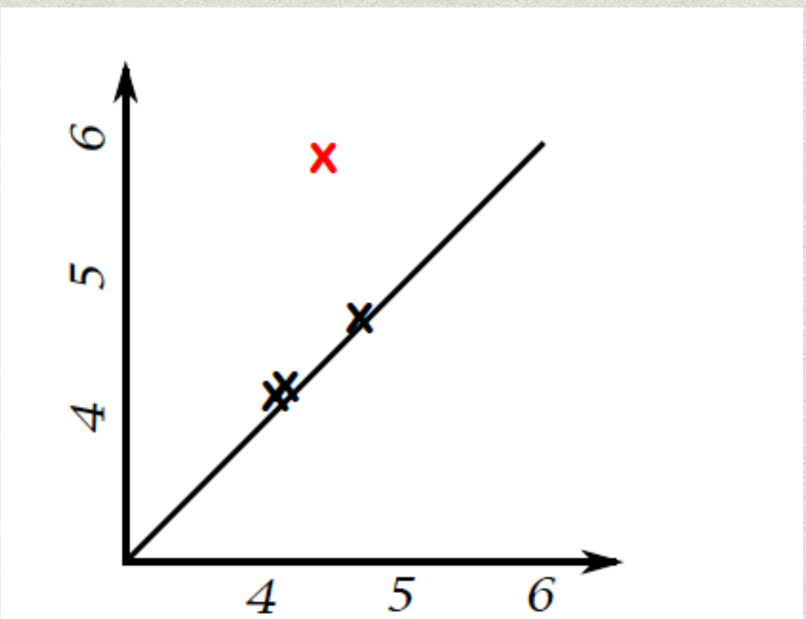

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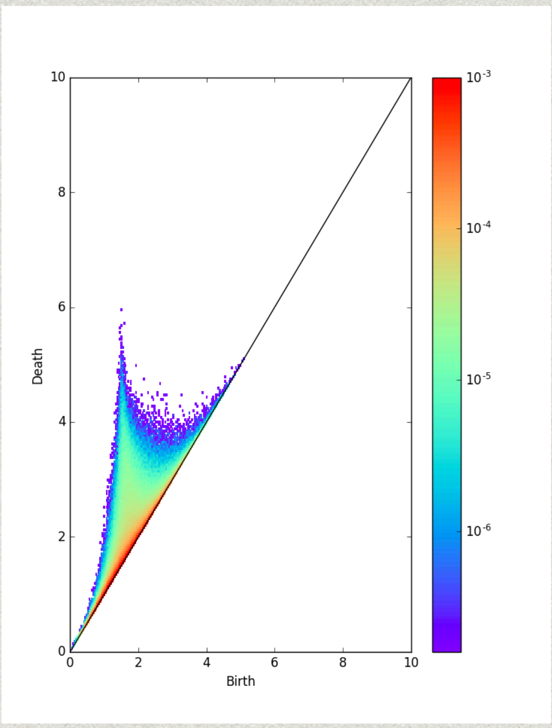

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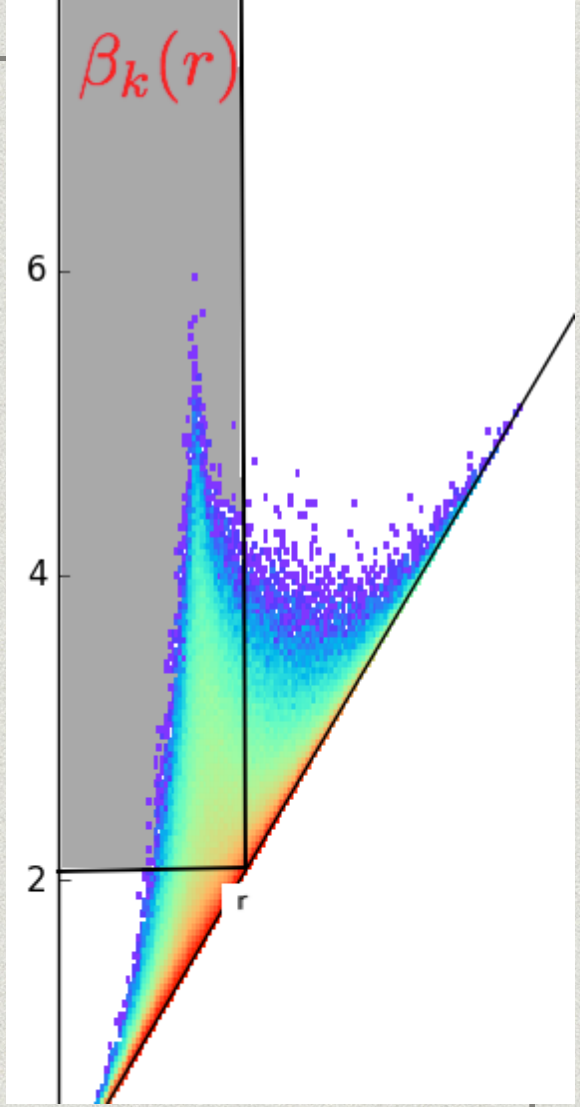

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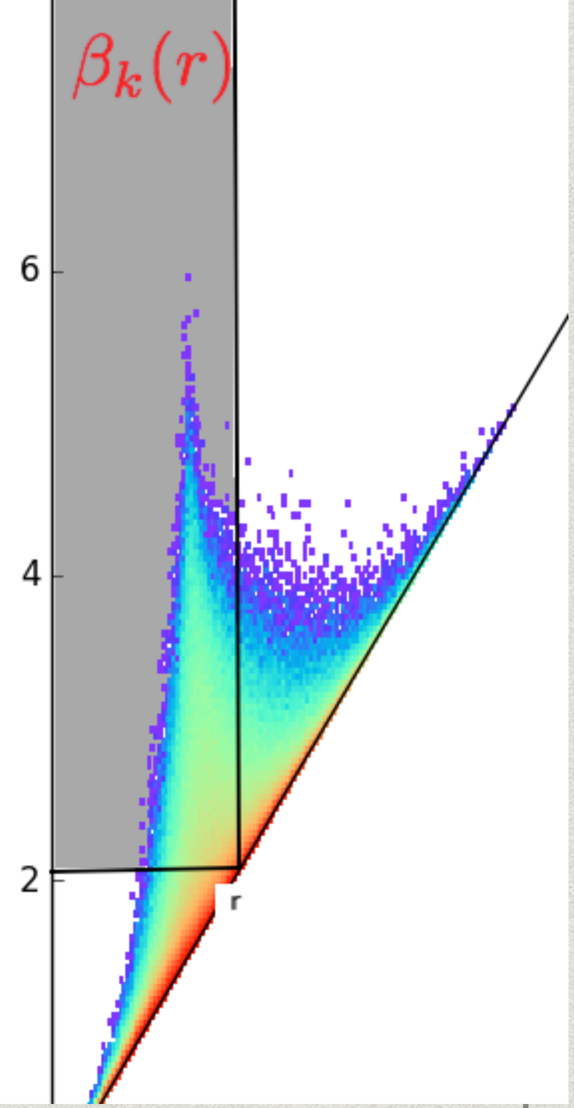

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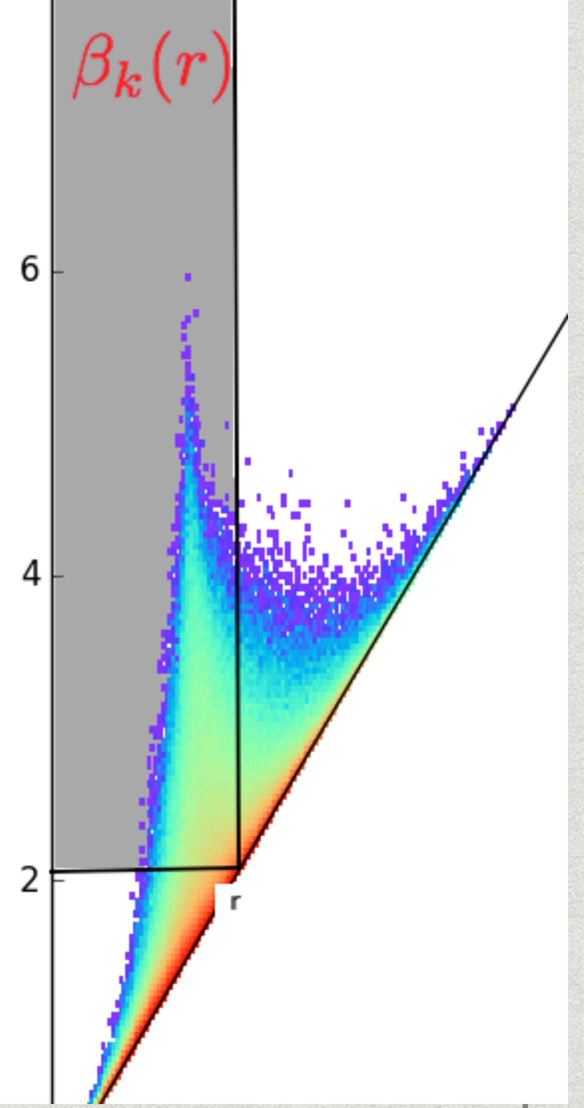

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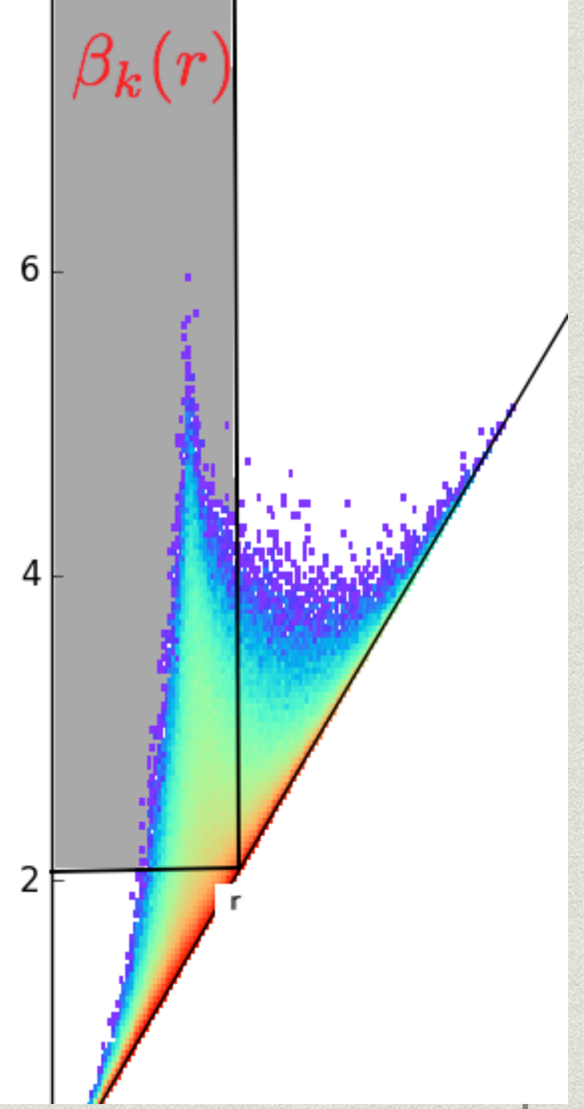

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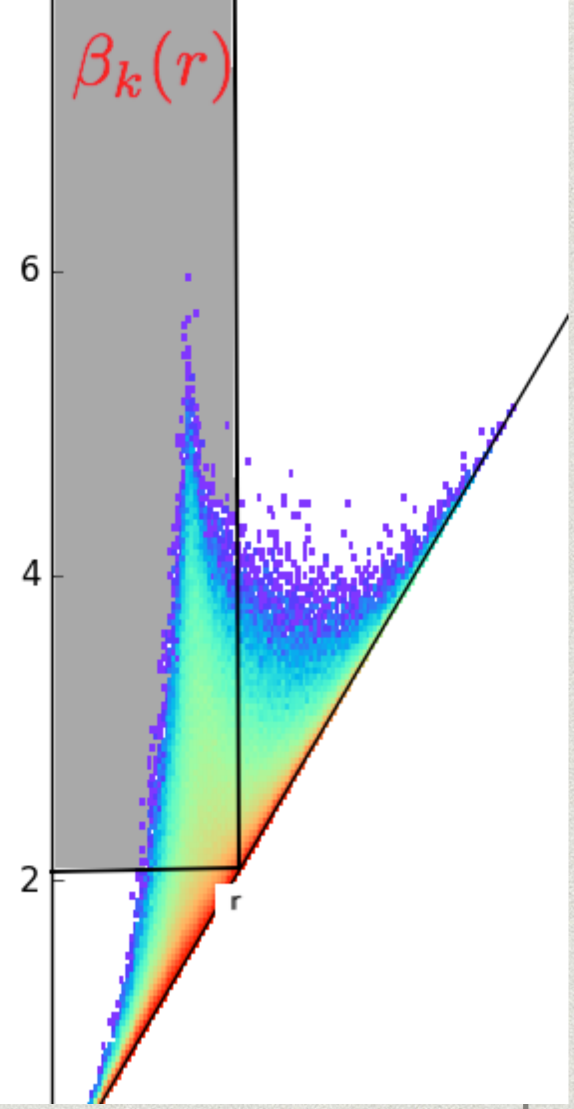

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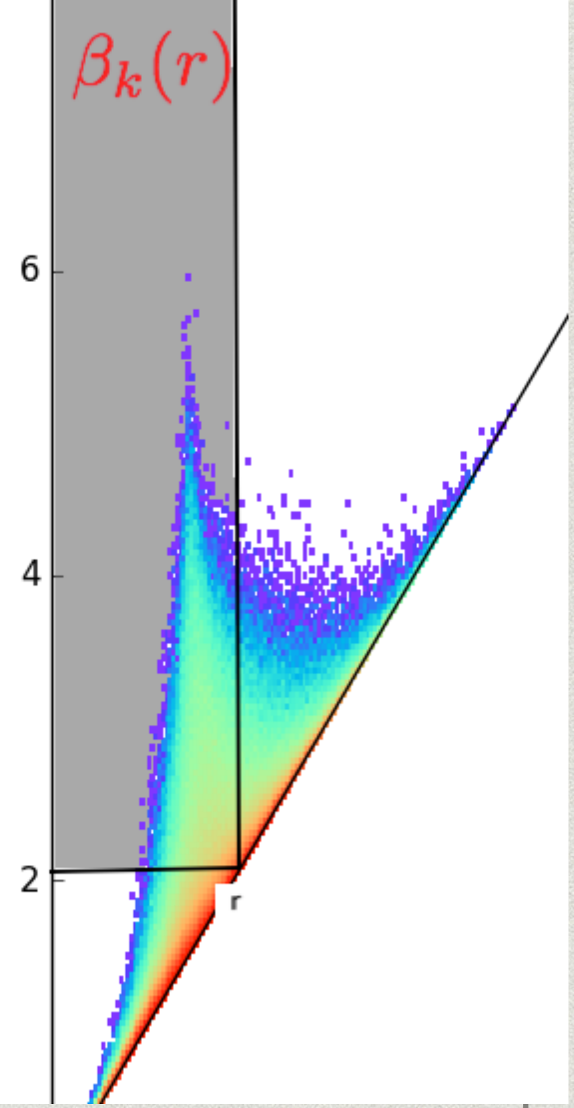

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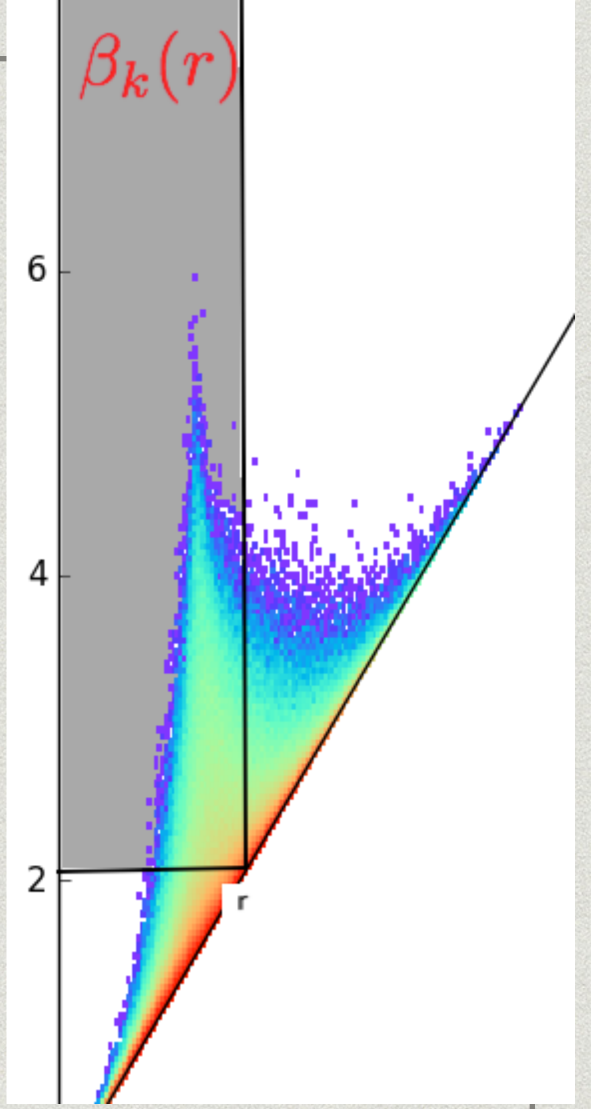

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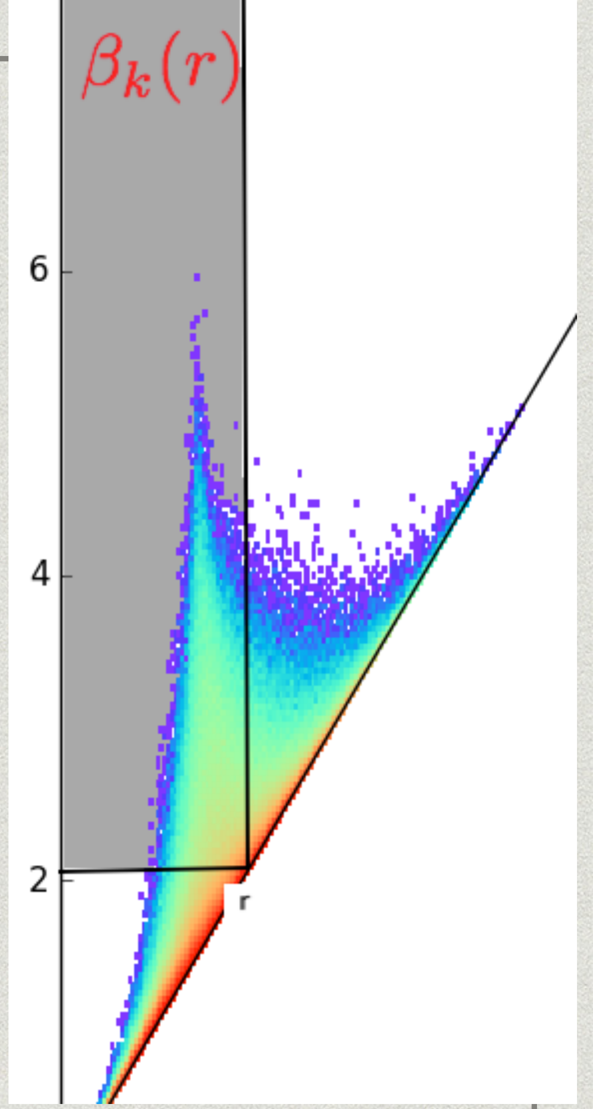

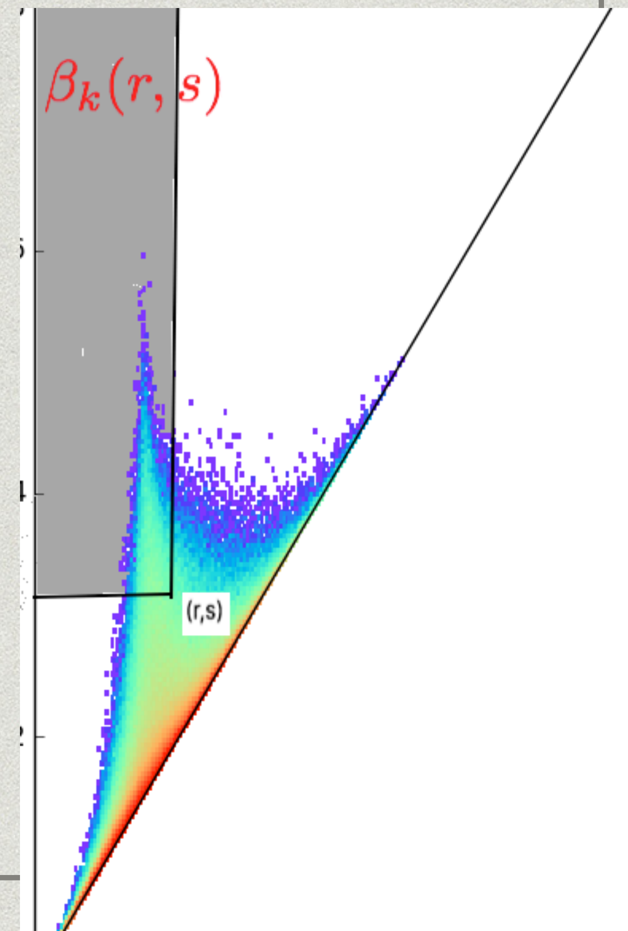

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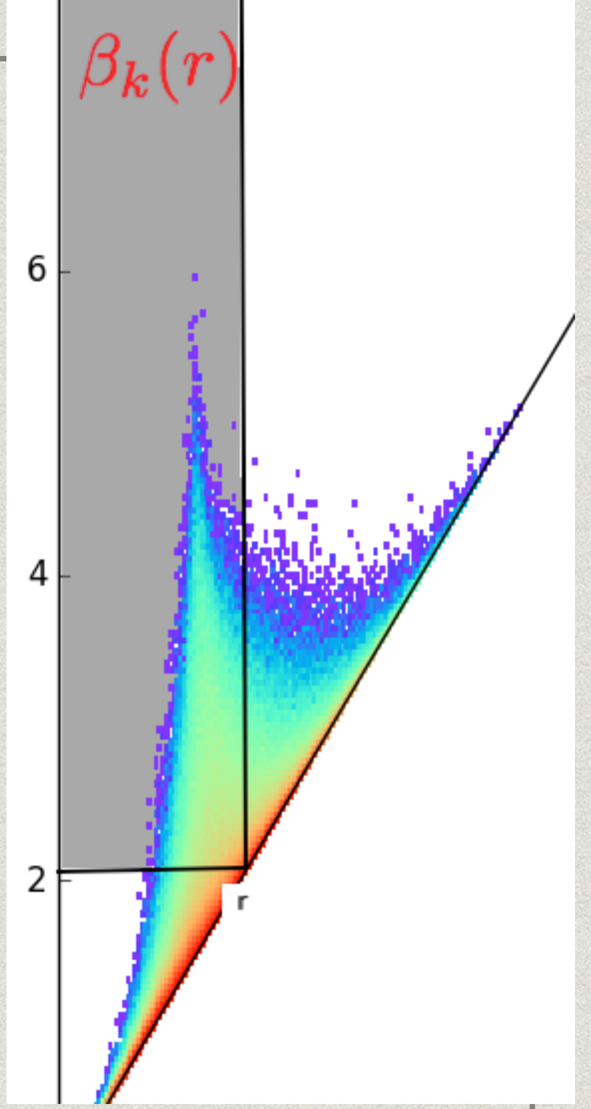

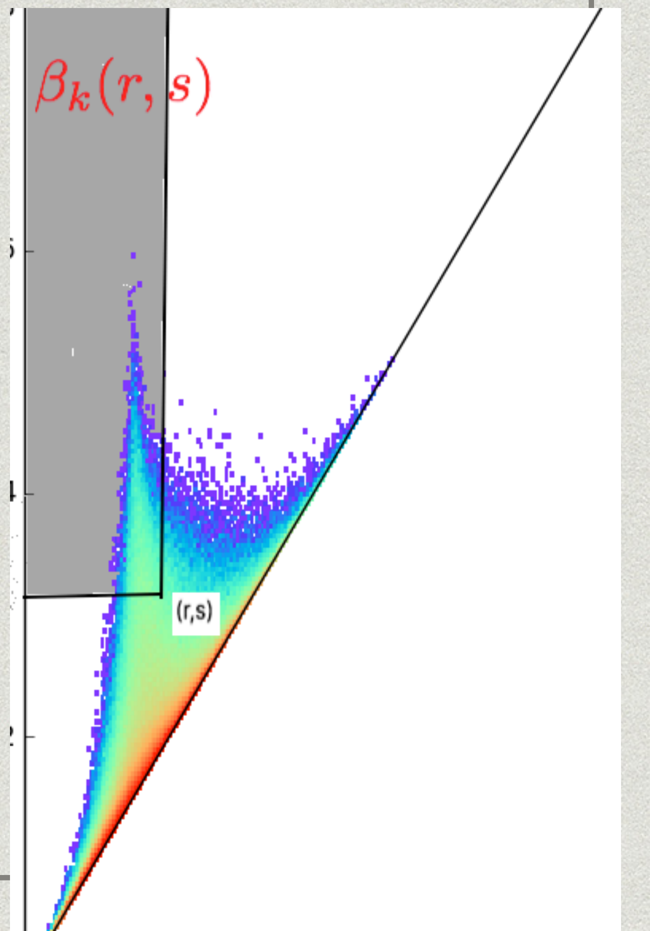

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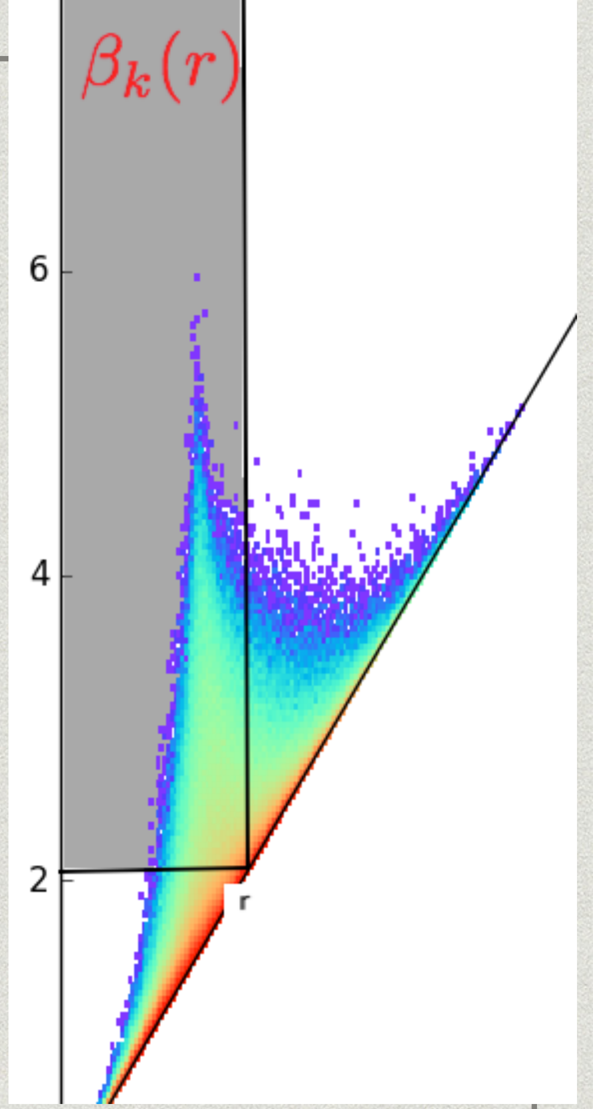

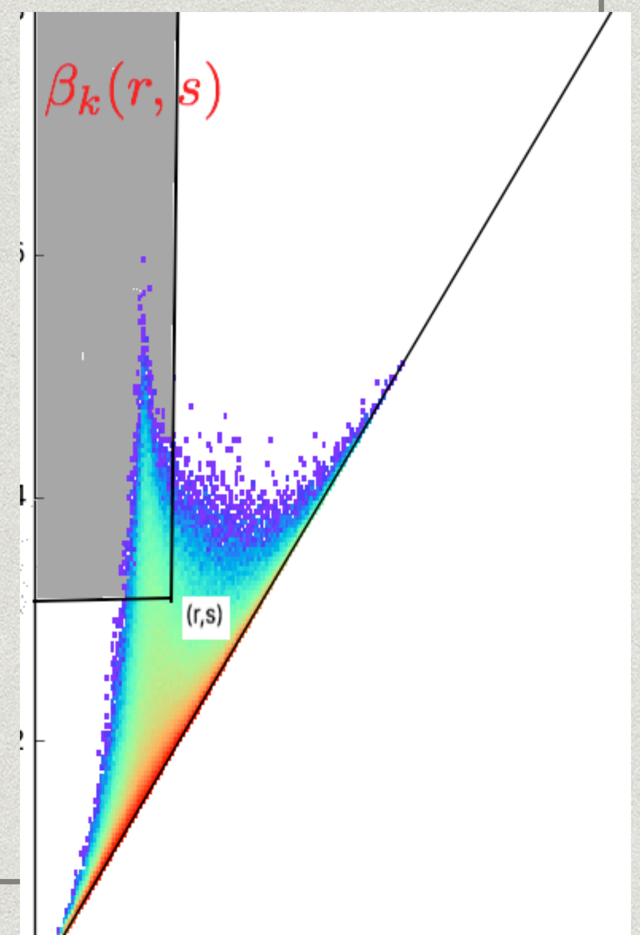

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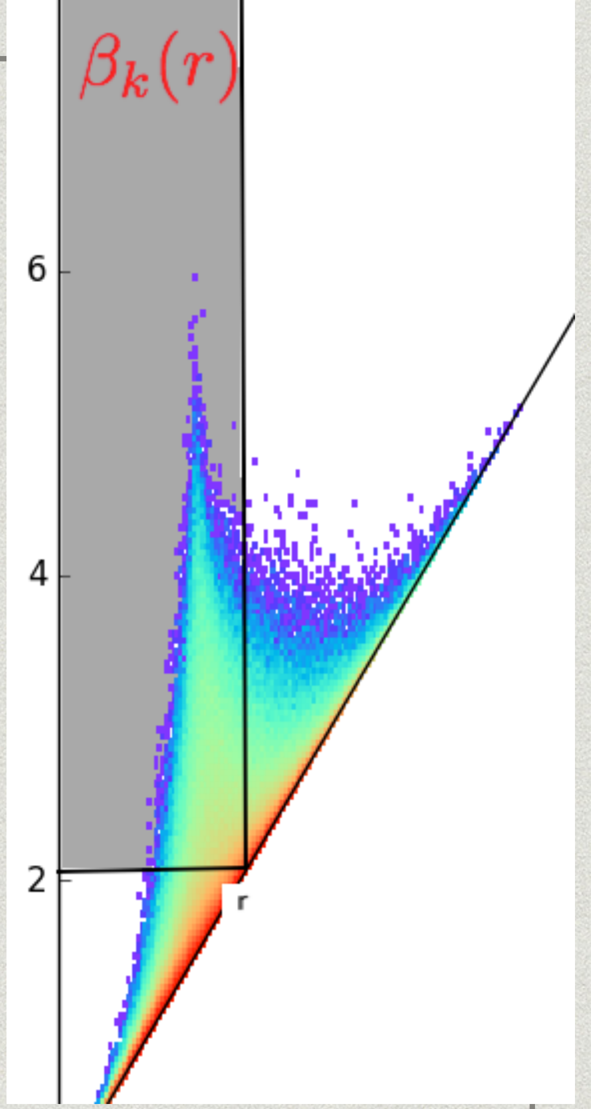

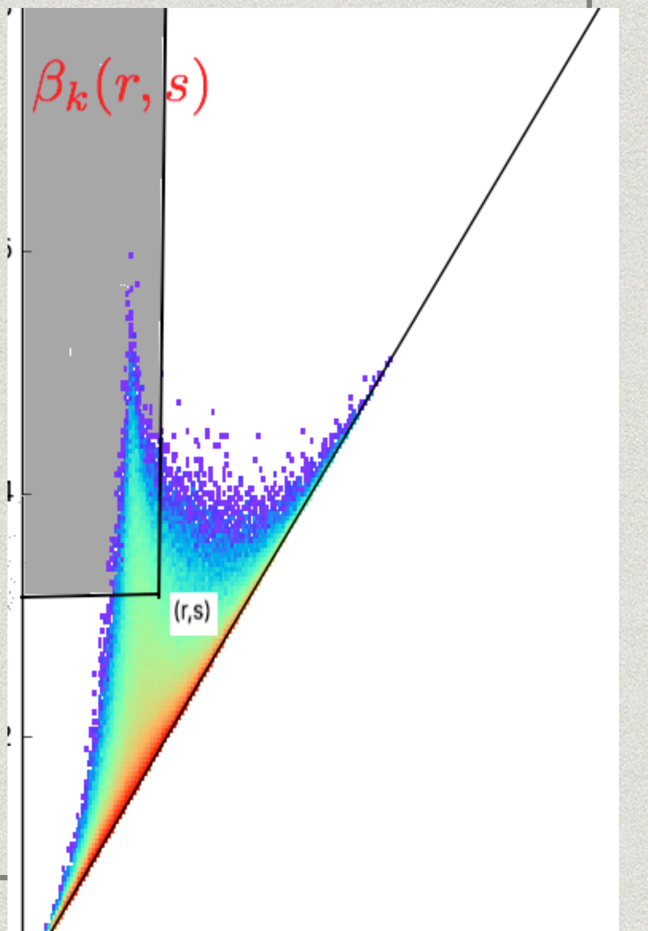

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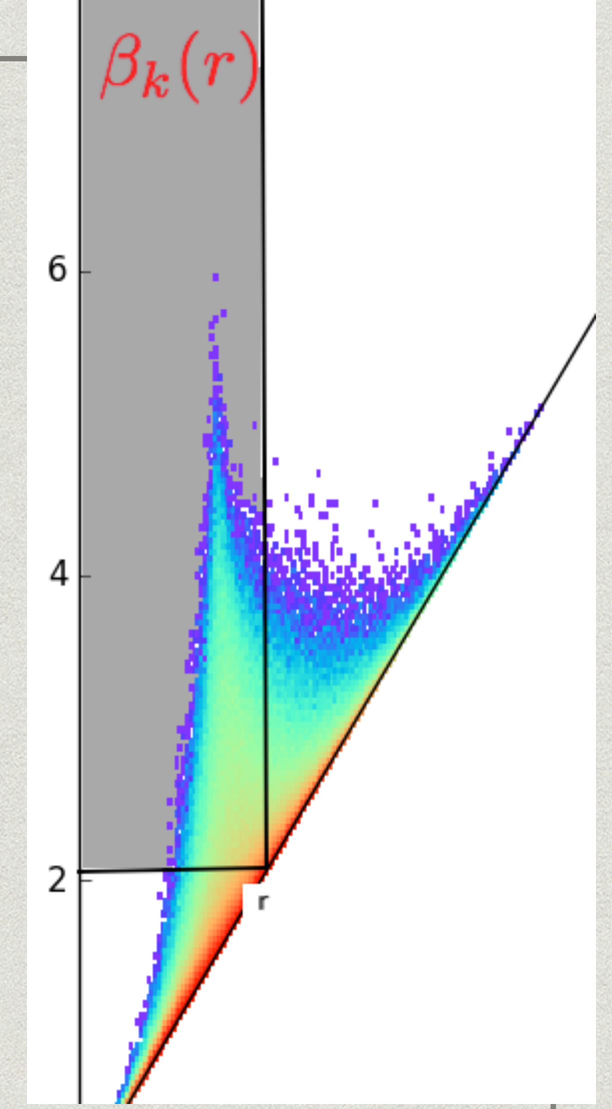

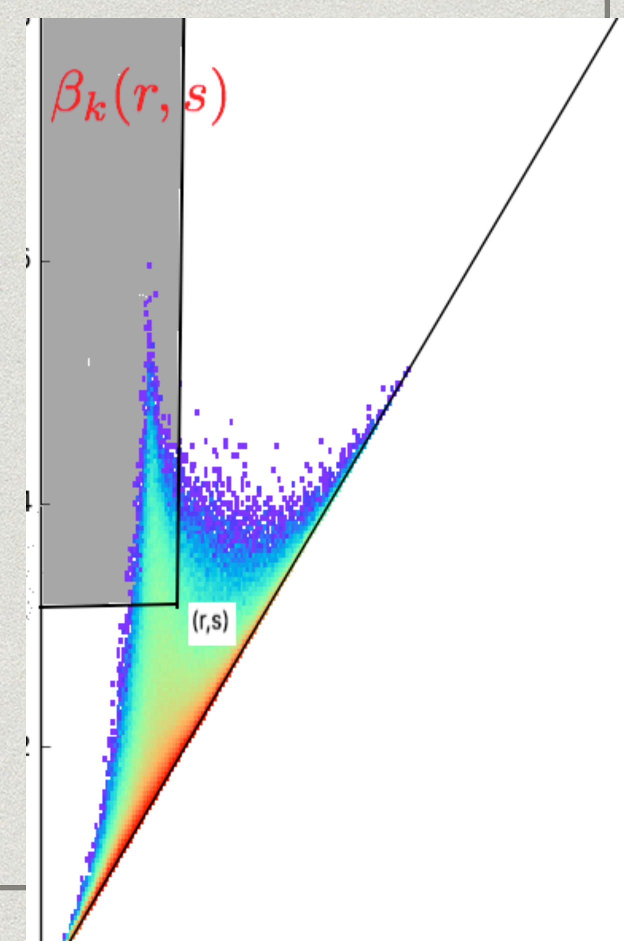

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F is translation invariant, apply Ergodic theorem to RHS and use the approximation.

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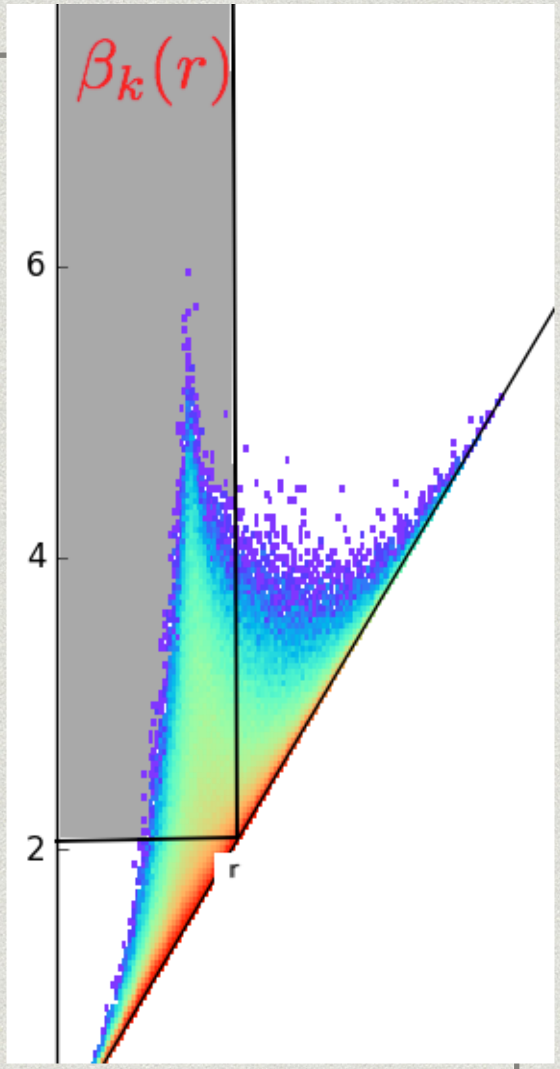

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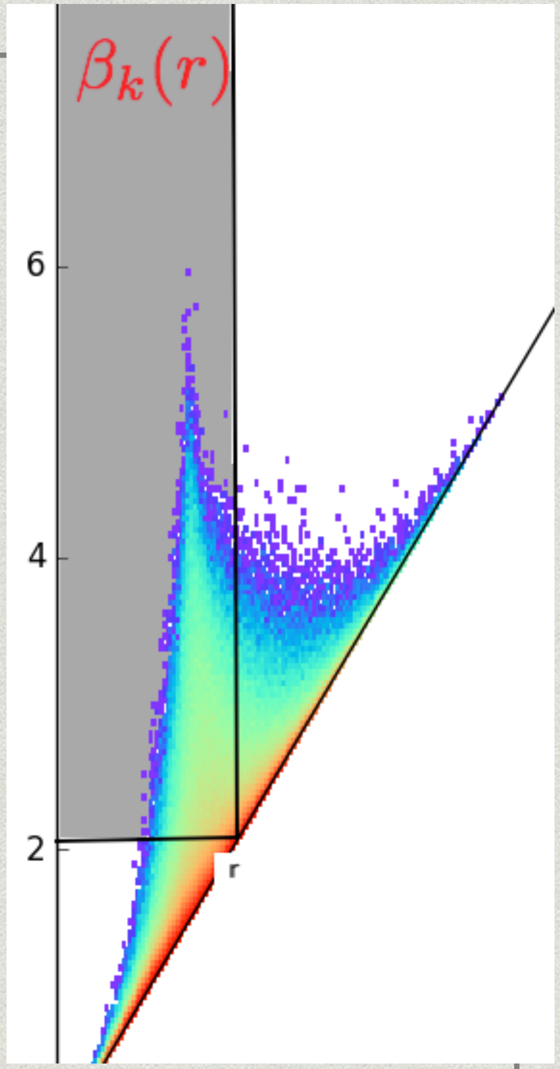

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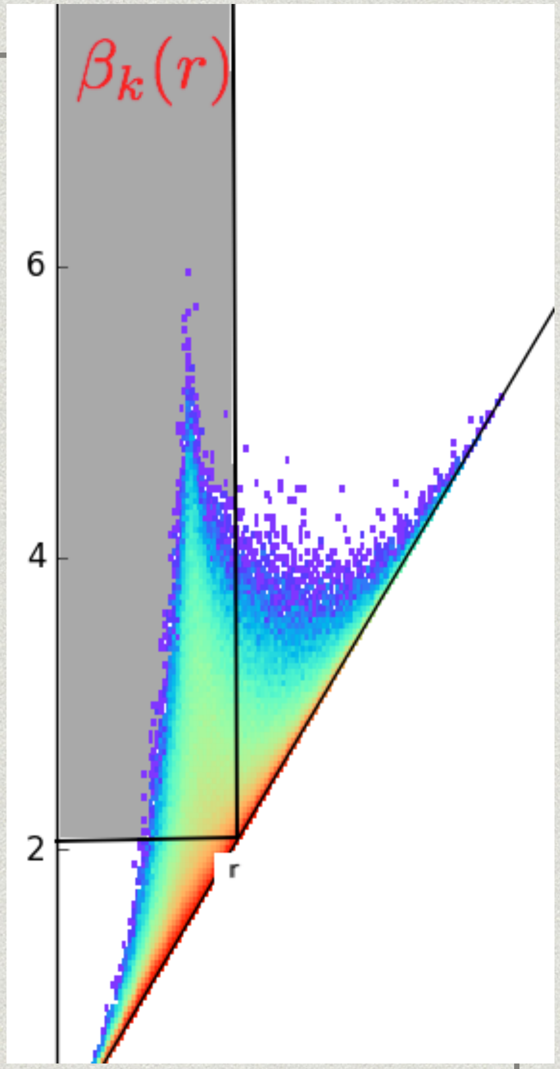

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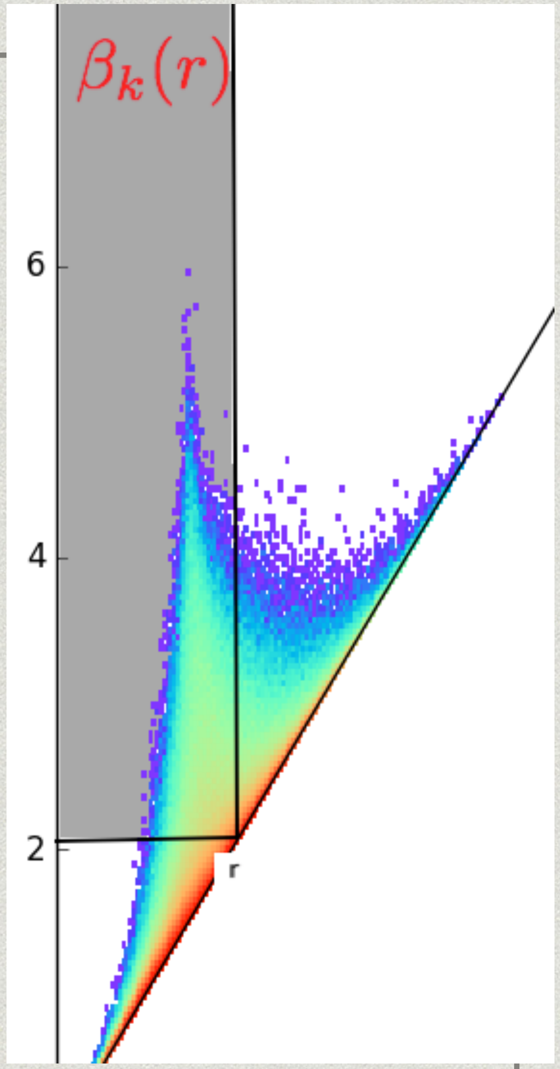

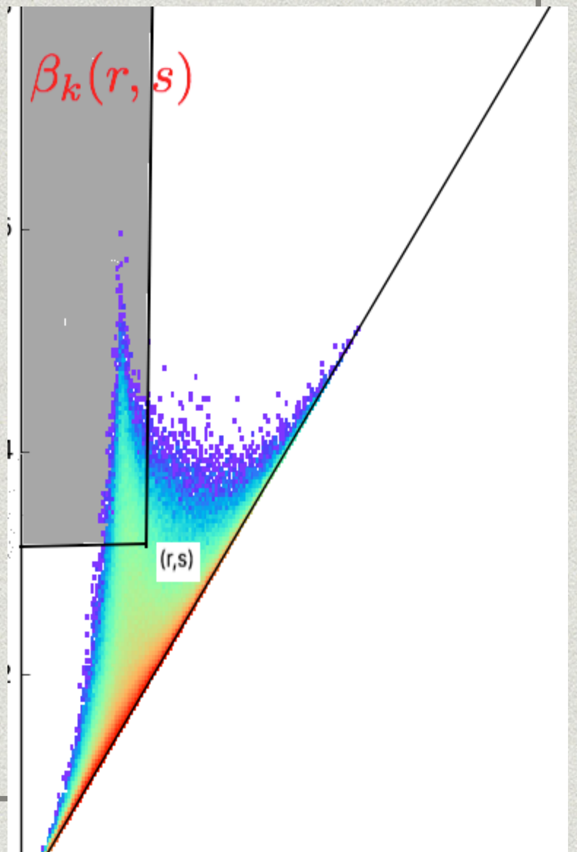

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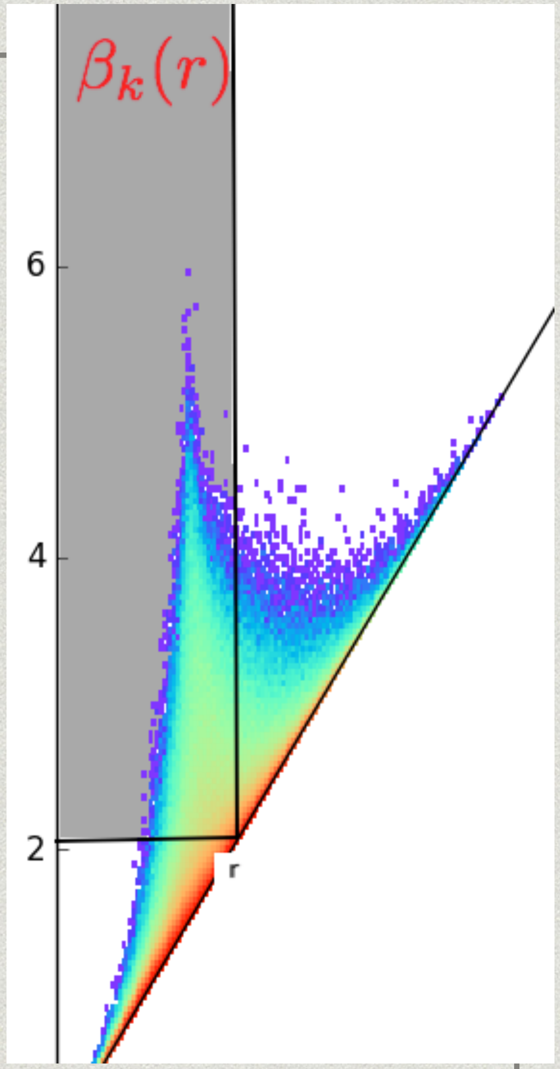

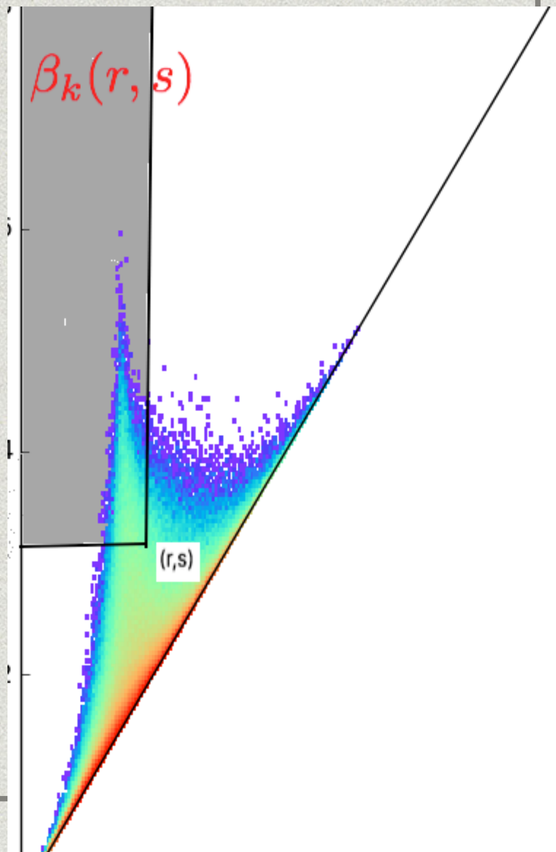

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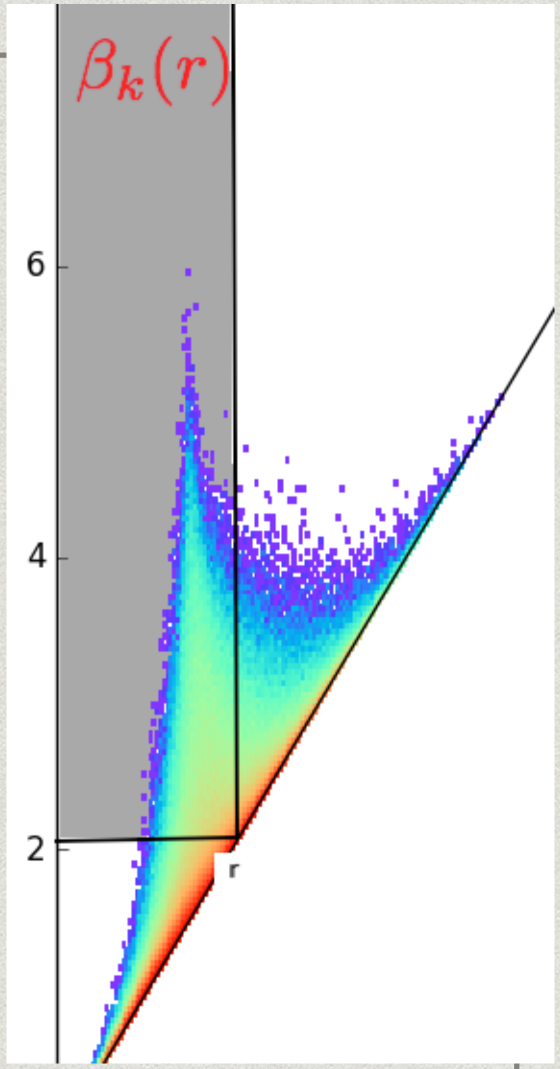

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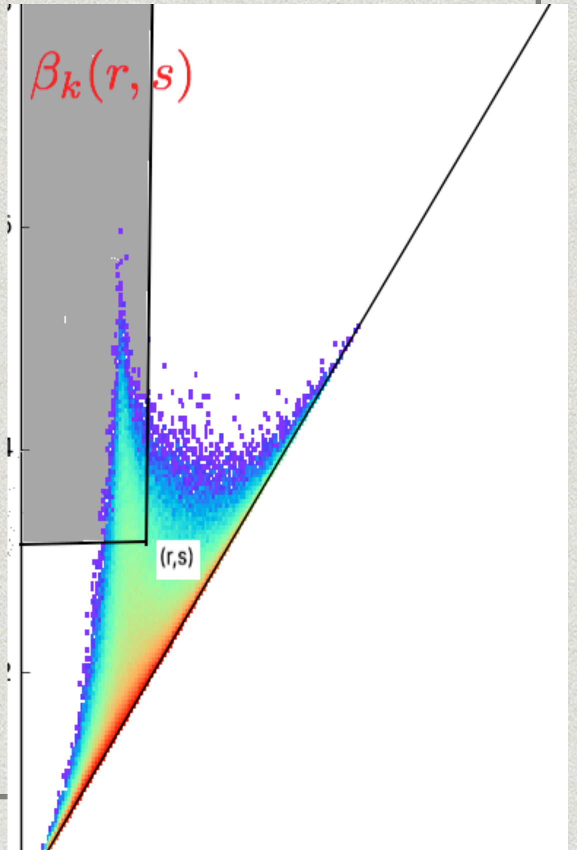

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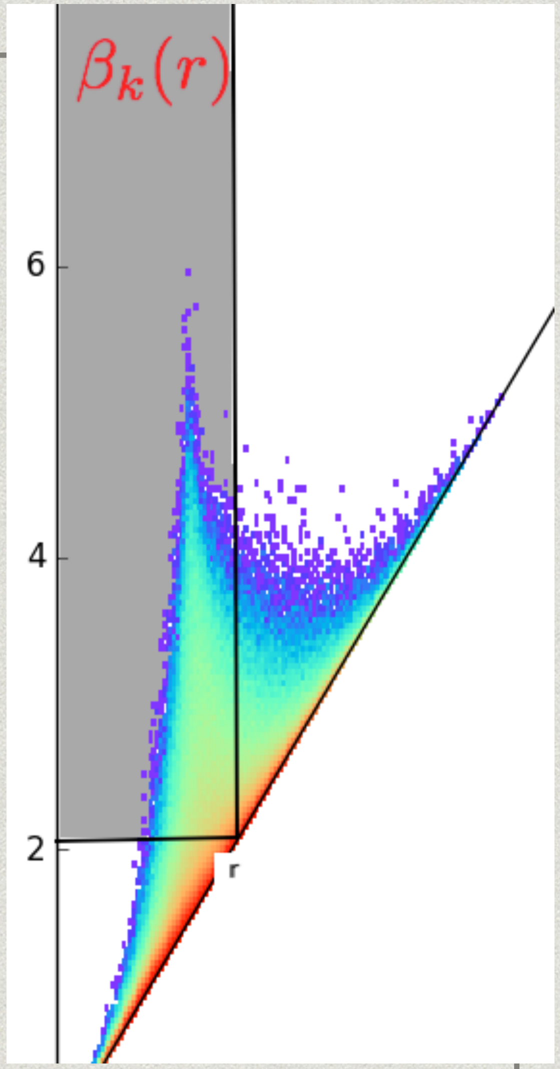

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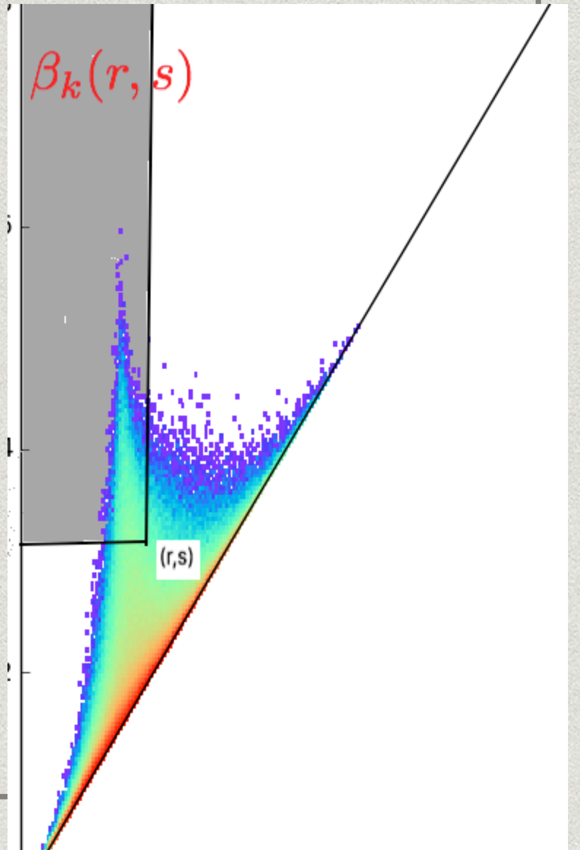

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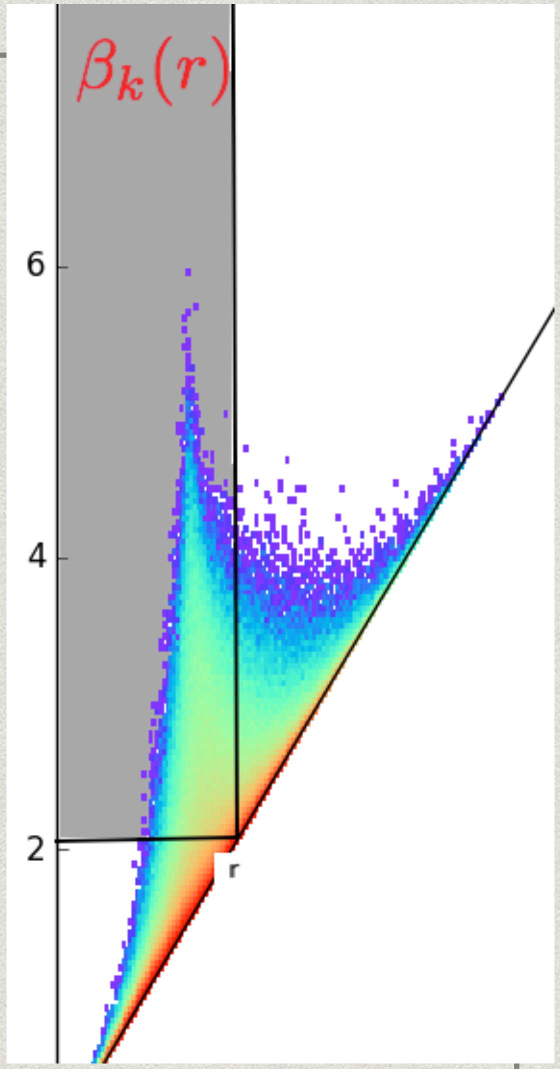

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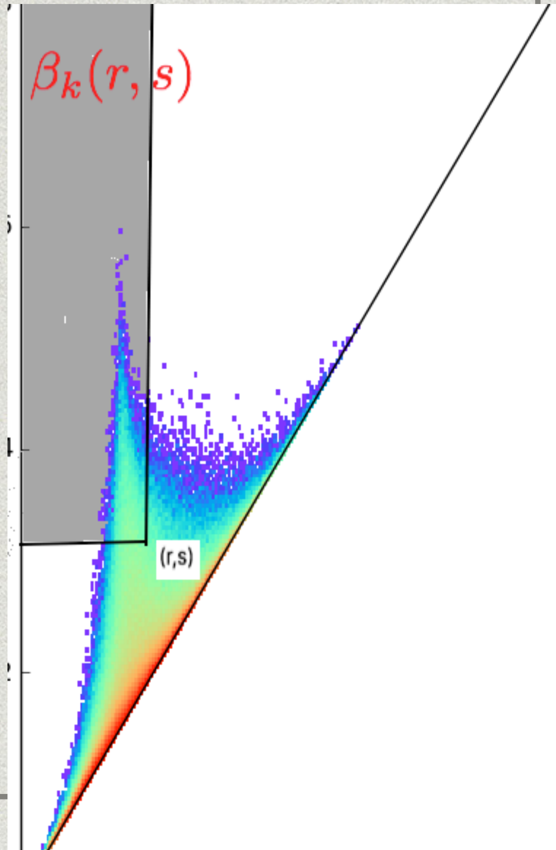

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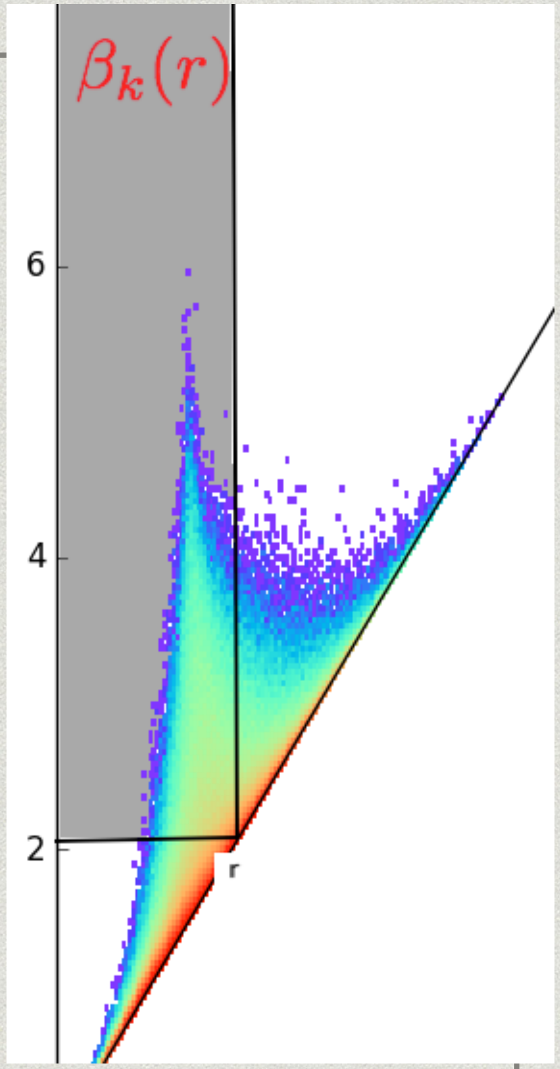

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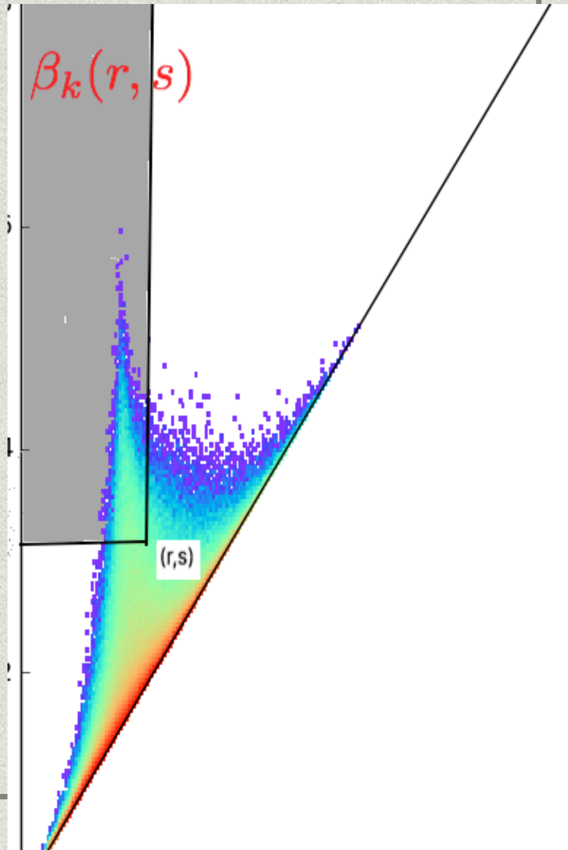

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For $\beta_k(r)$, use Mayer-Vietoris exact sequence, but rest involve more direct analysis.

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More on MSAs in **"A Probability Meeting to be Named Later"**, May 12-14, 2017, ISI Bangalore.

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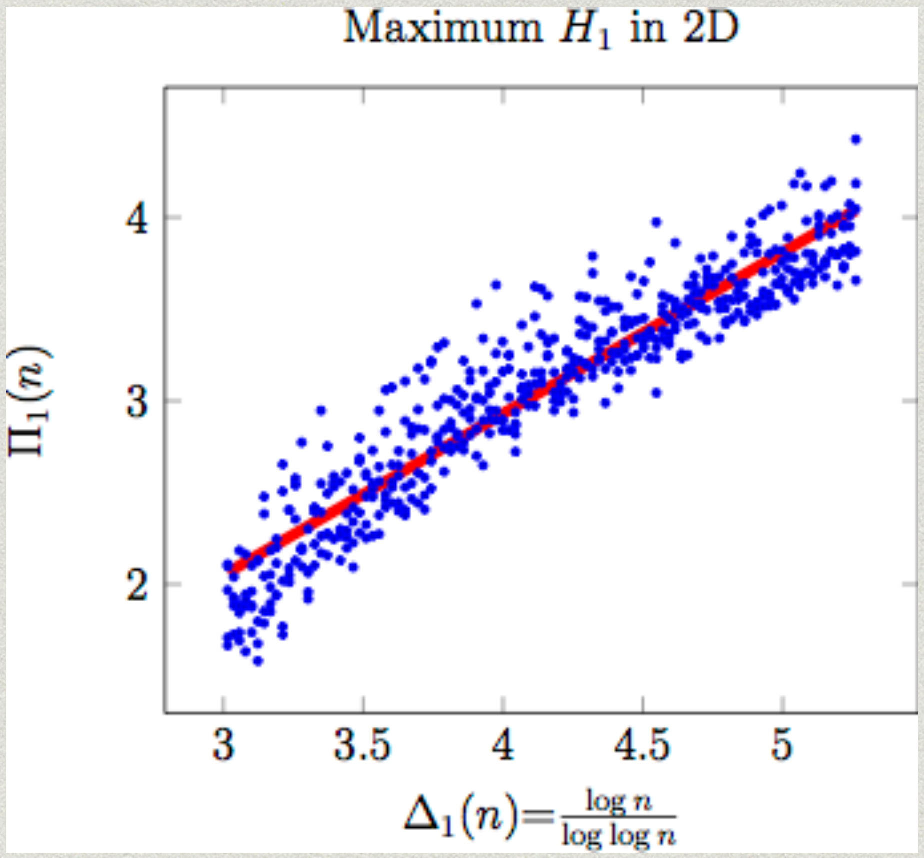

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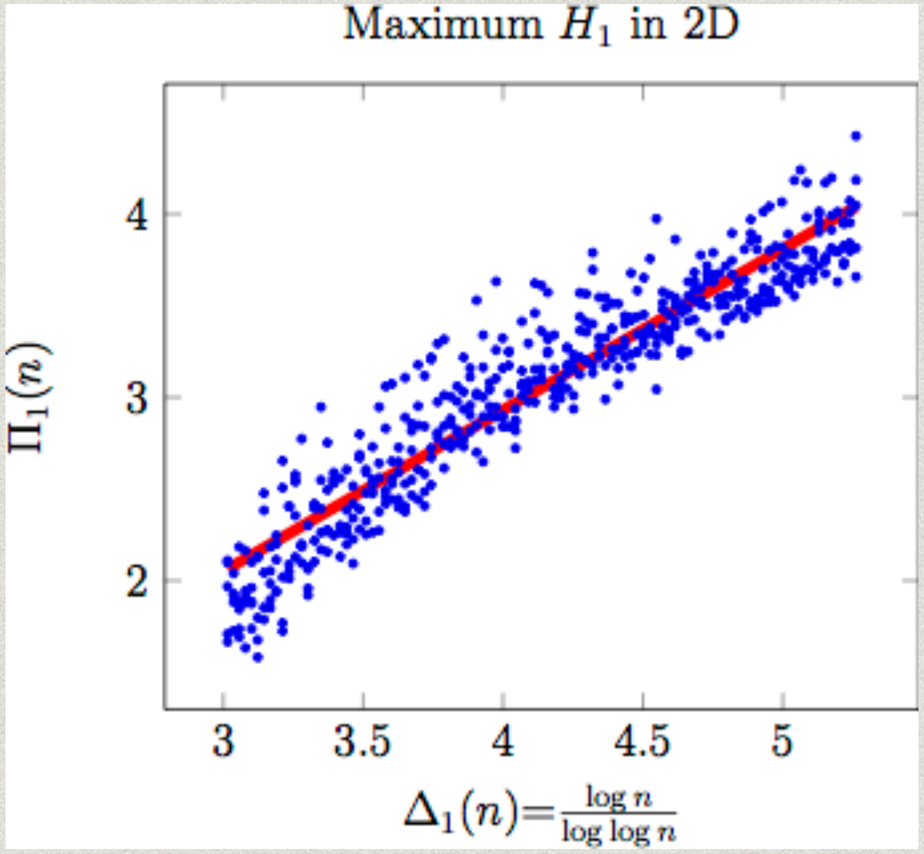

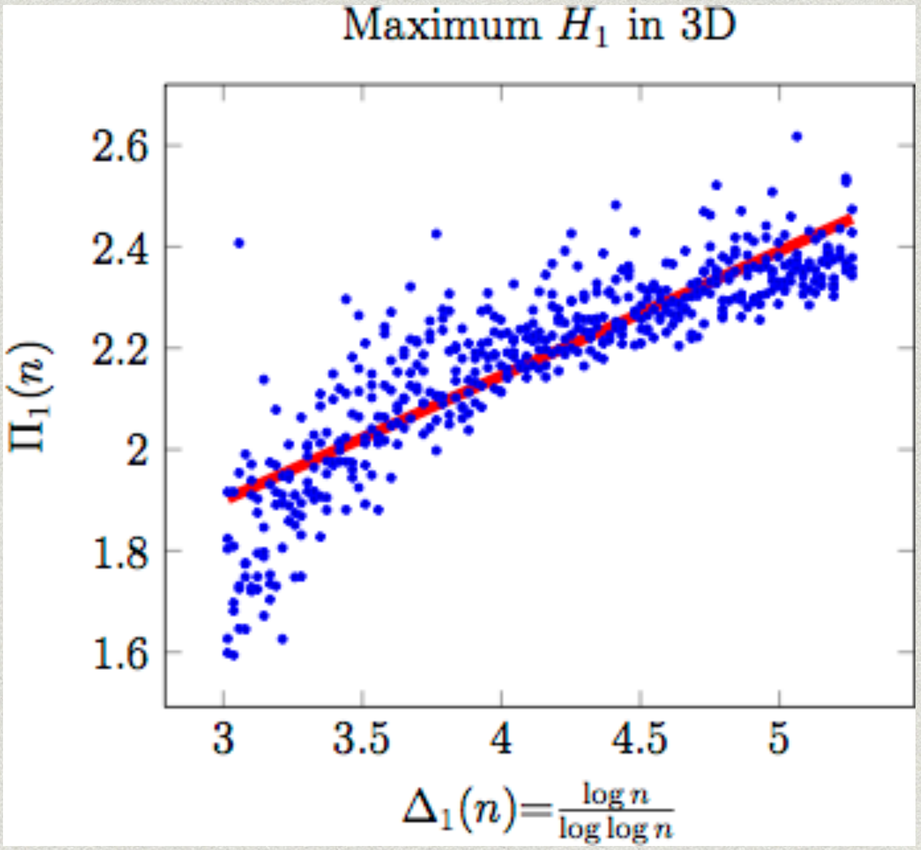

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Does $\chi(r), r \geq 0$ reveal a lot of information ?

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$\chi(r)$ has $(d-1)$ zeros in d-dimensions.

Zero (in r) of $\bar{\chi}(r)$ corresponds to percolation threshold in $d = 2$.

R. Neher, K. Mecke, H. Wagner, Topological estimation of percolation thresholds. J. Stat. Mech., 2008.

Does $\chi(r), r \geq 0$ reveal a lot of information ?

Robert J. Adler, TDA and the Euler Characteristic Curve**.** *IMA Talk, 2013.*

"I predict a new subject of statistical topology.
Rather than count the number of holes,
Betti numbers, etc., one will be more interested
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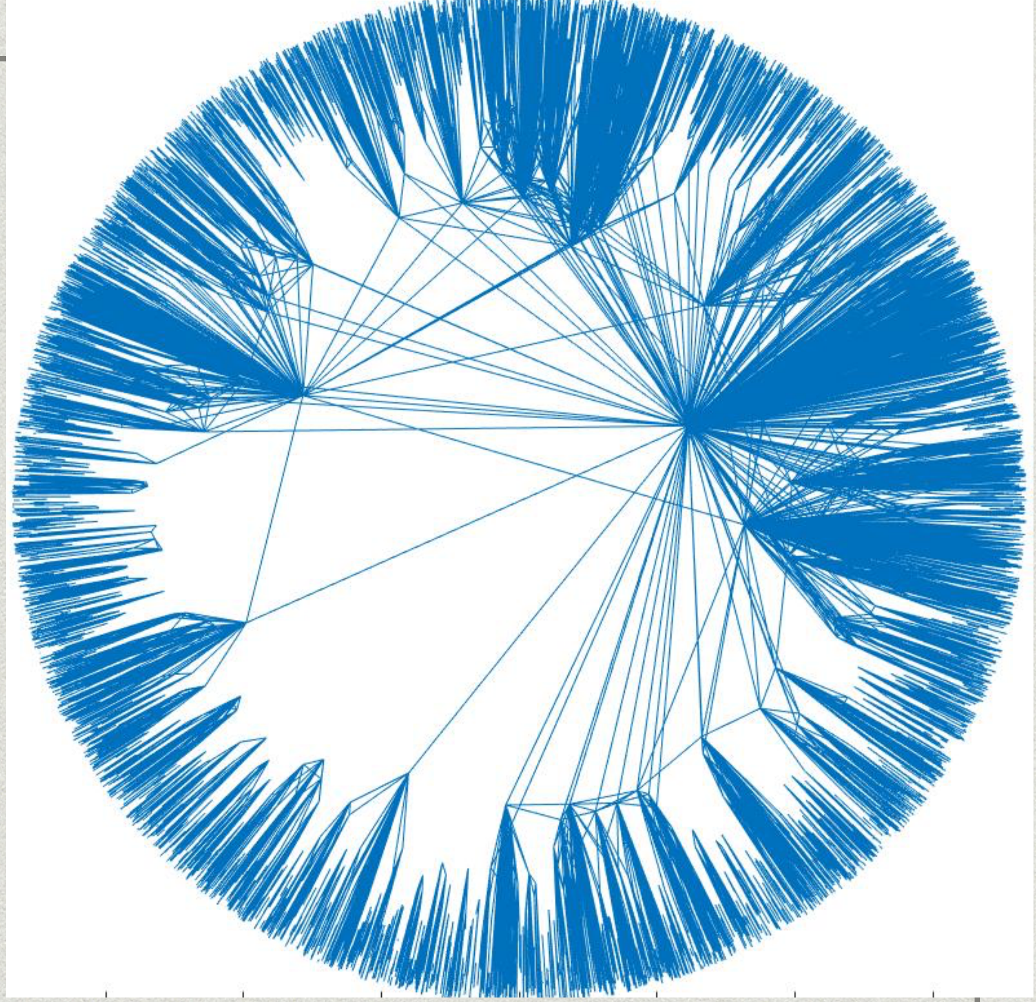

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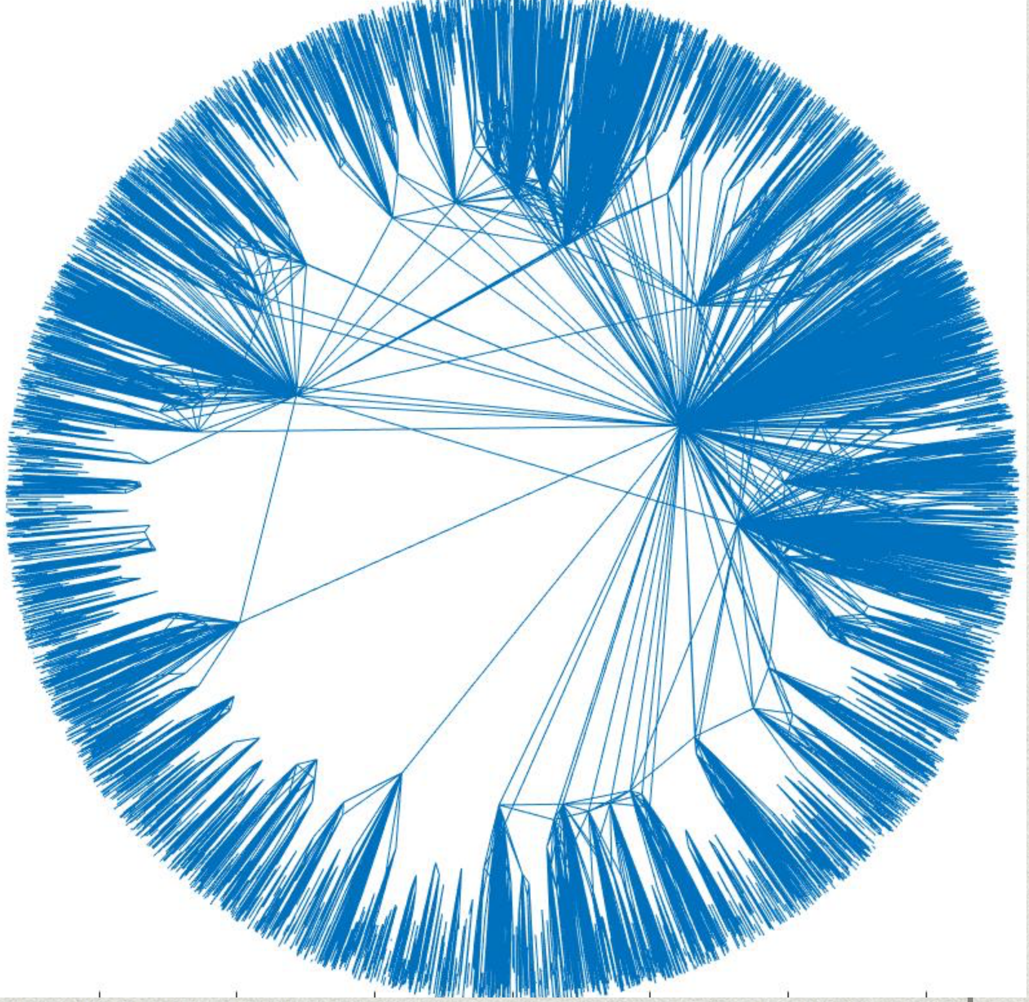

Robert J. Adler, TOPOS**.** *IMS Bulletin, 2016.*

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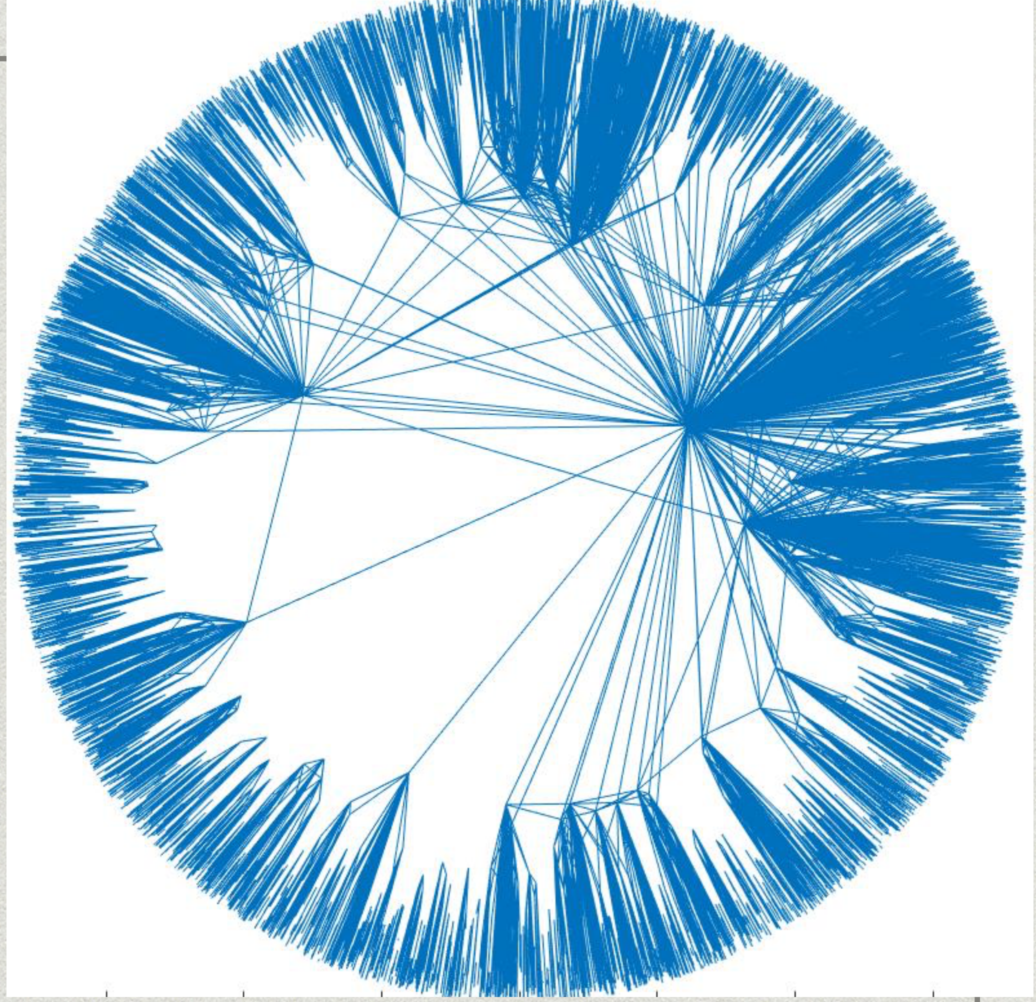

Robert J. Adler, TOPOS**.** *IMS Bulletin, 2016.*

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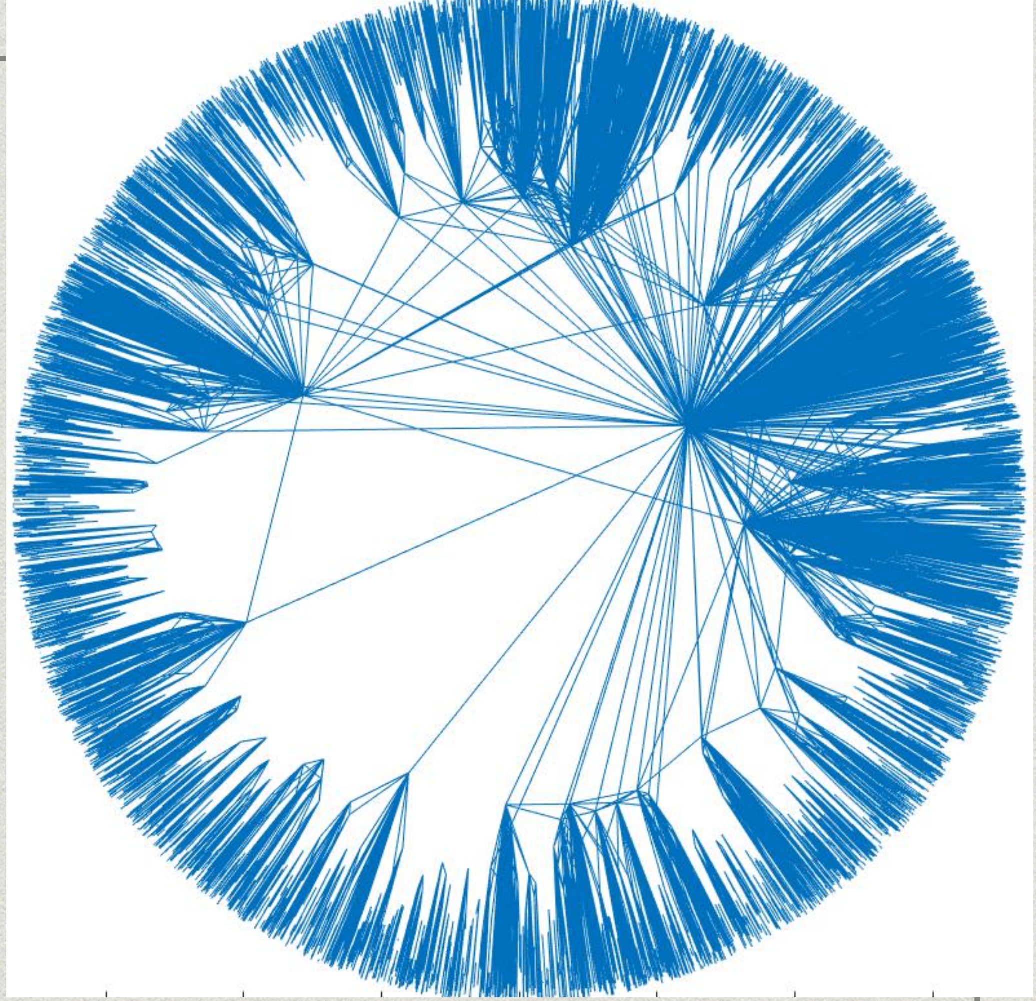

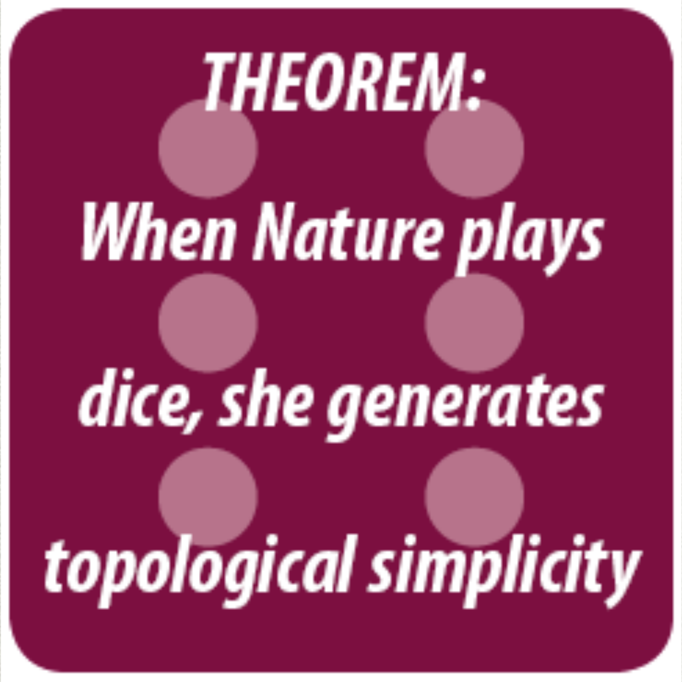

Robert J. Adler, TOPOS. *IMS Bulletin, 2016.*

Useful surveys :

1. **Gunnar Carlsson.** Topological pattern recognition for point cloud data. 2014.
2. **Mathew Kahle,** Topology of random simplicial complexes. 2014.
3. **Omer Bobrowski and Mathew Kahle,** Topology of random geometric complexes. 2015