Point Set Pattern Matching under Rigid Motion

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Outline

1 Introduction
2 Previous Work
3 Exact Point Set Pattern Matching
4 Approximate Point Set Pattern Matching
5 Translation
6 Rigid Motion
Introduction

A problem in pattern matching is to design efficient algorithms for testing how closely a query set $Q$ of $k$ points resembles a sample set $P$ of $n$ points, where $k \leq n$.

In image processing, computer vision and related applications like finger print matching, point sets represent some spatial features.

The problem has several variants [Alt and Guibas, 1999] based on:

- class of allowable transformations
- exact or approximate matching
- equal cardinality, subset, or largest common point set matching
# Problem Variants

## Allowable Transformations

- Translation
- Translation and Rotation - **Rigid Motion Transform**
- Translation, Rotation and Scaling - **Similarity Transform**

Distances are preserved in Translation and rigid motion.

## Types of Matching

- **Exact matching**: points in query set match exactly with points in sample set after the requisite transform.
- **Approximate matching**: points in query set after the requisite transform go to a **neighborhood** of points in the sample set.
Motivation

Reference fingerprint

Query fingerprint
Anchor

- Anchor centroids and match distances.
- But anchoring centroids do not help when we match $Q$ with a set of points $P' \subseteq P$. 

**Exact Matching under Rigid Motion**

$|P| = |Q|$

**Exact Matching**

$P$

$Q$

- centroid
Exact Matching under Rigid Motion

Anchor centroids and match distances.

But anchoring centroids do not help when we match $Q$ with a set of points $P' \subseteq P$. 
Exact Matching under Rigid Motion

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Approximate Partial Matching under Rigid Motion

Given two point sets $P$ and $Q$ ($|P| = |Q|$), check if there is a bijection $\ell : Q \rightarrow P$ and a congruence $T$, such that $T(q) \in U_\epsilon(\ell(q))$, $\forall q \in Q'$, where $Q' \subseteq Q$, and $U_\epsilon(p)$ denotes the closed $\epsilon$-neighborhood of a point $p \in P$. 

$|Q| \leq |P|$

$\epsilon$ neighborhoods may or may not overlap.
Outline

1. Introduction
2. Previous Work
3. Exact Point Set Pattern Matching
4. Approximate Point Set Pattern Matching
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Previous Work

Exact Matching

- Rezende and Lee (1995) proposed an $O(kn^d)$ time algorithm for partial point set pattern matching, where $d$ (* dimension of the plane *), $n = |P|$ (* size of sample set *), $k = |Q|$ (* size of pattern set *).
  It allows translation, rotation and scaling.

- Akutsu, Tamaki and Tokuyama (1998) proposed an $O(kn^{4/3} + A)$ time algorithm for testing the congruence in 2D, where $A =$ time complexity for locating $r$-th smallest distance among a set of $n$ points in 2D.
  $= O(n^{4/3} \log^{8/3} n)$ (on an average).

- Akutsu, Tamaki and Tokuyama (1998) proposed a Las Vegas expected time algorithm of time complexity $O(n^{4/3} \log^{8/3} n + \min(k^{0.77} n^{1.43} \log n, n^{4/3} k \log n))$ using parametric search.
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Approximate Matching (for $k = n$ case)

- Alt et al. (1988) designed a general algorithm of $O(n^8)$ time that works for overlapping and non-overlapping $\epsilon$-circles and $\epsilon$-boxes.
- They use a geometric fact and bipartite graph matching.
- A valid matching of $Q$ with $P$ exists iff there is a matching where $q_i, q_j \in Q$ are matched exactly to the boundaries of $U_\epsilon(p_\alpha), U_\epsilon(p_\beta)$ of two points $p_\alpha, p_\beta \in P$.
- The algorithms are of high time complexity and involve computing the intersection of complex algebraic curves.
- Heffernan and Schirra (1994) show that this $O(n^8)$ algorithm is indeed optimal for $\epsilon$-circles.
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Previous Work

Approximate Matching (for $k \neq n$ case)

Chew et al. (1997) proposed an $(n^2 k^3 \log^2 k n)$ time algorithm for approximate partial point set pattern matching under rigid motion.
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1. Introduction
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4. Approximate Point Set Pattern Matching
5. Translation
6. Rigid Motion
Exact Point Set Pattern Matching

The Problem

Input: \( P = \{ p_1, p_2, \ldots, p_n \} \) (* Sample Set *)
\( Q = \{ q_1, q_2, \ldots, q_k \} \) (* Query Set *),
\( k \leq n \).

Output: A subset of points in \( P \) that are matched with the points in \( Q \) if match exists under rigid motion.

Trivial Algorithm

Choose all possible \( \binom{n}{k} \) subset of \( P \) and test for a match under rigid motion.
Exact Point Set Pattern Matching

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Input: \[ P = \{ p_1, p_2, \ldots, p_n \} \text{ (* Sample Set *)} \]
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An obvious improvement

Preprocessing: Use circular sorting to create a data structure attached with each point in $P$.

**Space:** $O(n^2)$

**Time:** $O(n^2)$.

Query:
- Sort the query point set angularly.
- Anchor a pair of query point with each pair of sample point.
- It determines the rotation angle and scaling.
- Match the other query points with a subset of sample points.

**Time:** $O(k \log k + kn(n - 1))$
Exact Point Set Pattern Matching

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Demonstration
An Efficient Algorithm for Rigid Motion

Fact 1

All the $\binom{k}{2}$ distances must occur in the $\binom{n}{2}$ distances in $P$

Fact 2

[Szekely 1997] In a sample set $P$ of size $n$, the maximum number of equidistant pairs of points is $O(n^{4/3})$ in the worst case.
An Efficient Algorithm for Rigid Motion

**Fact 1**

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An Efficient Algorithm for Rigid Motion

Preprocessing

- Sort the \( \binom{n}{2} \) distances of \( P \) distances in \( P \).
- Create a height balanced binary tree \( T \) with distinct distances.
- Attach an array \( \chi_\delta \) with each element \( \delta \) of the tree. Its each element is a triple \( (p_i, p_j, \psi_{ij}) \), where \( \psi_{ij} \) is the angle of the line \( (p_i, p_j) \) with x-axis.
- Each point \( p_i \in P \) is attached with a height-balanced binary tree \( S_i \). Its elements are tuples \( (r_{ij}, \theta_{ij}) \).

Time Complexity: \( O(n^2 \log n) \)
Space Complexity: \( O(n^2) \)
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Time Complexity: \( O(n^2 \log n) \)

Space Complexity: \( O(n^2) \)
An Efficient Algorithm for Rigid Motion

**Query**

- Take two points \((q_1, q_2)\), and check whether \(\lambda(q_1, q_2) \in T\). If not, report no match found.
- Let \(\lambda(q_1, q_2) = \delta\).
- We consider each member \(\lambda(p_i, p_j) \in \chi\delta\).
- Anchor \((q_1, q_2)\) with \((p_i, p_j)\), and search in \(S_i\) for the presence of a match.

**Time Complexity**

For each \(\lambda(p_i, p_j) = \delta\), searching for a match needs \(O(k \log n)\) time. So,

Time Complexity: \(O(n^{\frac{4}{3}} k \log n)\)
An Efficient Algorithm for Rigid Motion

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Time Complexity: \(O(n^\frac{4}{3} k \log n)\)
An Important Note

Though the worst case number of equidistant pairs in a point set of size $n$ is $O(n^{\frac{4}{3}})$, in a random instance actually the number is very less.

<table>
<thead>
<tr>
<th>Number of points (n)</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lceil n^{\frac{4}{3}} \rceil$</td>
<td>465</td>
<td>1170</td>
<td>3969</td>
<td>10001</td>
</tr>
<tr>
<td>Maximum number of Equidistant pairs observed</td>
<td>4</td>
<td>6</td>
<td>20</td>
<td>53</td>
</tr>
</tbody>
</table>
## Experimental Results

<table>
<thead>
<tr>
<th>No. of points in sample</th>
<th>No. of points in query</th>
<th>No. of anchoring</th>
<th>CPU time for Rezende and Lee</th>
<th>CPU time for our Algorithm</th>
<th>% savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
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<td>1</td>
<td>730.0</td>
<td>22.0</td>
<td>97.0</td>
</tr>
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<td>23.4</td>
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<td>1905.0</td>
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<td>32.0</td>
<td>98.0</td>
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<td>129.0</td>
<td>97.0</td>
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<td>76.0</td>
<td>98.0</td>
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<tr>
<td>800</td>
<td>11</td>
<td>4098.0</td>
<td></td>
<td>11.0</td>
<td>97.0</td>
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</table>
Outline

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2. Previous Work
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4. Approximate Point Set Pattern Matching
5. Translation
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Our Effort

We assume that

- the $\epsilon$-neighborhoods are axis-parallel squares of side length $\epsilon$.

- $P$ is well separated, i.e. each pair of points $p, p' \in P$ satisfy either $|x(p) - x(p')| \geq \epsilon$ or $|y(p) - y(p')| \geq 3\epsilon$ or both.
Our Effort

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Approximate Point Set Pattern Matching

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A Necessary Characterization for a Matching

by Lemma 1 and 2

zone_{ep}
Lemma (an iff condition)

If $\exists$ a transformation $T(Q)$ for the said match, then $\exists$ another transformation $T'(Q)$, such that one point of $Q$ lies on the left boundary of the $\epsilon$-box of a point in $P$, and one point of $Q$ lies on the top boundary of the $\epsilon$-box of a point in $P$.

Definition

Consider an $\epsilon$-box $ABCD$ around $p \in P$. The extended $\epsilon$-box of $p$ is a $\epsilon \times 2\epsilon$ box formed by attaching another $\epsilon \times \epsilon$ square $CDFE$ above the $\epsilon$-box $ABCD$. $CDFE$ is called the extended portion of the $\epsilon$-box.
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Lemma (an if condition)

If \( \exists \) a transformation \( T(Q) \) for the said match, \( \exists \) another transformation \( T'(Q) \), such that

- a point \( q \in Q \) lies at the top-left corner of the \( \epsilon \)-box of a point in \( P \).
- at least one point lies in the extended portion of the \( \epsilon \)-box
- each of the remaining members in \( Q \) lie in the extended \( \epsilon \)-box
Anchorings for Finding the Transformation

Anchoring for Translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

- Two unknown parameters, $t_x$ and $t_y$. So, one anchoring is needed.

Anchoring for Rigid Motion

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

- Three unknown parameters, $t_x$, $t_y$ and $\theta$. So, two anchorings are needed.
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Matching under Translation

by Lemma 1 and 2

\{ zone ep \}

\{ zone eb \}

\{ zone \}
Translation

The Algorithm

- Position $q$ at the top-left corner of the $\epsilon$-box of $p$, and check whether each point in $Q \setminus \{q\}$ lies inside the extended box of some point of $P$.
- If the above checking returns false, then no match exists with $q$ at the top-left corner of the $\epsilon$-box of $p$.
- If it returns true, then the points in $Q \setminus \{q\}$ can be partitioned into $Q_1$ and $Q_2$. Each $q_1 \in Q_1$ lies in the $\epsilon$-box of some point in $P$. Each $q_2 \in Q_2$ lies in the extended portion of $\epsilon$-box of some member in $P$.
- Push points in $Q_2$ down such that points in $Q_1$ do not go out.
Translation

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The Analysis

The Algorithm

- Maintain a planar straight line graph (PSLG) data structure with the $\epsilon$-boxes around each point in $P$.
- Anchoring a point $q \in Q$ with the top-left corner of an $\epsilon$-box, we perform $k$ point location queries in the PSLG. This needs $O(k \log n)$ time.
- Pushing points down take another $O(k)$ time.
- There can be $O(nk)$ anchorings in the worst case.

Theorem

The worst case time complexity of the approximate matching of $Q$ with a $k$-subset of $P$ in 2D when only translation is considered is $O(nk^2 \log n)$. 
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- Anchoring a point $q \in Q$ with the top-left corner of an $\epsilon$-box, we perform $k$ point location queries in the PSLG. This needs $O(k \log n)$ time.
- Pushing points down take another $O(k)$ time.
- There can be $O(nk)$ anchorings in the worst case.

Theorem

The worst case time complexity of the approximate matching of $Q$ with a $k$-subset of $P$ in 2D when only translation is considered is $O(nk^2 \log n)$. 
### The Analysis

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Outline

1. Introduction
2. Previous Work
3. Exact Point Set Pattern Matching
4. Approximate Point Set Pattern Matching
5. Translation
6. Rigid Motion
Matching under Rigid Motion
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- Consider the $k - 1$ concentric circles $C_{ij} \forall q_j \in Q \setminus \{q_i\}$. Each circle $C_{ij}$ intersects some extended $\epsilon$-boxes.

- As $P$ is well-separated, these intersections contribute a set of non-overlapping arcs that define a circular arc graph $G$.

- Each circle $C_{ij}$ may intersect $O(n)$ boxes making $O(nk)$ nodes in $G$. So, there can be $O(nk)$ cliques of size $k - 1$.

- Each clique $\chi$ corresponding to an anchoring of $q$ represents an angular interval $I^* = [\theta_1^*, \theta_2^*]$ such that all points in $Q \setminus q$ lie in some extended $\epsilon$-box.
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One of the $k - 1$ concentric circles shown.
Processing Each Clique

Processing Cliques

- Each clique $\chi$ represents an angular interval (an arc) $I^* = [\theta_1^*, \theta_2^*]$. For any angle of rotation $\theta \in I^*$, each of the $k - 1$ points of $Q \setminus q_i$ lies inside $k - 1$ disjoint extended $\epsilon$-boxes.

- Partition the arcs corresponding to the points in $Q \setminus \{q_i\}$ into two subsets $Q_1$ and $Q_2$. Arcs in $Q_1$ are all inside the $\epsilon$-boxes, and those in $Q_2$ are all inside the extended portion of the $\epsilon$-boxes.

- Now we need to push down the points so that a match, if it exists, can be found.

- We do this for all arcs together.
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The Pushing Down

![Diagram](image-url)
Homogeneous Splitting

Homogeneous splitting of $I^*$

- Consider a $\theta \in I^*$.
- For each $q \in Q_1$, let $f_q^1(\theta)$ denotes the distance of $q$ from the bottom of the corresponding $\epsilon$-box.
- For each $q \in Q_2$, $f_q^2(\theta)$ denotes the distance of $q$ from the top of the corresponding $\epsilon$-box.
- The functions $f_q^i(\theta) i = 1, 2$ are like $\overline{qaq} \sin(\alpha + \theta) - c$.

Observation

For a given $q \in Q_i$, the above functions are univariate, continuous, and unimodular.
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For a given $q \in Q_i$, the above functions are univariate, continuous, and unimodular.
# Envelope of Functions

**Definition ($\mathcal{L}(\theta)$ for $\theta \in \mathcal{I}^*$)**

$\mathcal{L}(\theta)$ denotes the lower envelope of $|Q_1|$ functions, namely $f_q^1(\theta)$, $q \in Q_1$.

**Definition ($\mathcal{U}(\theta)$ for $\theta \in \mathcal{I}^*$)**

$\mathcal{U}(\theta)$ denotes the upper envelope of $|Q_2|$ functions, namely $f_q^2(\theta)$, $q \in Q_2$. 

![Diagram of envelopes](image-url)
Envelope of Functions

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Envelopes of Functions

**Minimum Amount of Downward Translation for $Q_2$**

At a rotation angle $\theta \in \mathcal{I}^*$, the minimum amount of downward translation required to place the points in $Q_2$ in the corresponding $\epsilon$-box is $\max_{q \in Q_2} f^2_q(\theta) = \mathcal{U}(\theta)$.

**Maximum Amount of Downward Translation for $Q_1$**

Similarly, the maximum amount of downward translation that may retain all the points in $Q_1$ in its corresponding $\epsilon$-box is $\min_{q \in Q_1} f^1_q(\theta) = \mathcal{L}(\theta)$. 
Envelope of Functions

Minimum Amount of Downward Translation for $Q_2$
At a rotation angle $\theta \in \mathcal{I}^*$, the minimum amount of downward translation required to place the points in $Q_2$ in the corresponding $\epsilon$-box is $\max_{q \in Q_2} f^2_q(\theta) = U(\theta)$.

Maximum Amount of Downward Translation for $Q_1$
Similarly, the maximum amount of downward translation that may retain all the points in $Q_1$ in its corresponding $\epsilon$-box is $\min_{q \in Q_1} f^1_q(\theta) = L(\theta)$. 
**Definition**

A rotation angle $\theta$ is said to be a break-point in $\mathcal{L}$ if $\mathcal{L}(\theta) = f_{q_a}^1(\theta) = f_{q_b}^1(\theta)$ for $q_a, q_b \in Q_1$, $q_a \neq q_b$. Similarly, the break-points of the $\mathcal{U}$ function is defined.

**Lemma**

A pair of functions $f_{q'}^1(\theta)$ and $f_{q''}^1(\theta)$ (corresponding to $q', q'' \in Q_1$, $q' \neq q''$) w.r.t. $\theta \in \mathcal{I}^*$ may intersect in at most two points. The same is true for a pair of functions $f_{q'}^2(\theta)$ and $f_{q''}^2(\theta)$ for a pair of points $q', q'' \in Q_2$. 
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Observation

The collection of functions \( \{ f_q^1(\theta), q \in Q_1 \} \) follows a \((k, 2)\)-Davenport-Schinzel sequence. The same result is true for the collection of functions \( \{ f_q^2(\theta), q \in Q_2 \} \).

Lemma

The maximum number of break-points in the function \( \mathcal{L}(\theta) \) is \( \lambda_2(|Q_1|) = 2|Q_1| - 1 \), and it can be computed in \( O(|Q_1| \log |Q_1|) \) time. Similarly, for \( \mathcal{U}(\theta) \).

Moral of Homogeneous Splitting of \( I^* \)

\( I^* = [\theta_1^*, \theta_2^*] \) is split into \( O(k) \) sub-intervals defined by break-points of \( \mathcal{L}(\theta) \) and \( \mathcal{U}(\theta) \).
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Definition (Critical Angle)

If a match is found by a rotation of $Q$ by the angle $\theta^* \in I^*$ and a vertical downward shift, then $\theta^*$ is said to be a critical angle.

Figure: Envelopes and break points.
Computation of Critical Angle

Critical Angle Computation

For $\theta \in [\theta_1, \theta_2]$, the vertical downward shift will be determined by two points, $q_a, q_b \in Q$, where $q_a$ and $q_b$ are such that $f_1(q_a(\theta)) = L(\theta)$ and $f_2(q_b(\theta)) = U(\theta)$. 
Computation of Critical Angle

Critical Angle Computation

$\Delta_1 =$ Minimum amount of downward shift required to bring $q_a$ inside the $\epsilon$-box of $p'$. Thus,
$$\Delta_1 = \delta(q_i, q_\alpha)\sin(\theta + \theta_1) - (y(p') + \frac{\epsilon}{2}).$$

$\Delta_2 =$ Maximum amount of permissible downward shift keeping $q_b$ inside the $\epsilon$-box of $p''$. Thus,
$$\Delta_2 = \delta(q_i, q_\beta)\sin(\theta + \theta_1 + \psi) - (y(p'') - \frac{\epsilon}{2}).$$

A feasible solution $\theta$ (if it exists) must satisfy $\Delta_1 \leq \Delta_2$. 
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Complexity Analysis

Each point of $Q$ needs to be anchored at the top-left corner of the $\epsilon$-box of each point in $P$.

The nodes of the circular arc graph $G$ are obtained in $O(nk)$ time and the cliques of $G$ can be obtained in $O(nk\log n)$ time.

While processing a clique, computation of functions $U$ and $L$ needs $O(k\log k)$ time.

Number of elements in $U \cup L$ is $O(k)$ in the worst case.

The algebraic computation for processing each element in $U \cup L$ needs $O(1)$ time.
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The Final Result

**Theorem**

The time complexity of the proposed algorithm for \( \epsilon \)-approximate matching of \( Q \) with a subset of \( P \) where the neighbourhood around a point (in \( P \)) is defined as an \( \epsilon \)-box, is \( O(n^2k^2(\log n + k \log k)) \).
Further reading I


Thank You!