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Part I : Bases

- ordering and lattices
- erosion and dilation
- opening and closing

Jean Serra

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Image Processing

Image processing addresses three types of questions :

- *Codification ,*
- *Feature Extraction ,*
- *Segmentation .*

Image Processing

1- *Codification :*

It comprises all modes of representation. In particular:

Acquisition:
analog => digital

Compression:
change in the representation.

Synthesis:
new image from numbers.

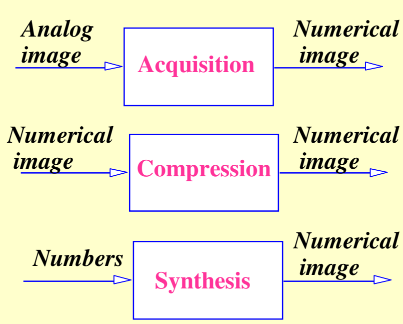
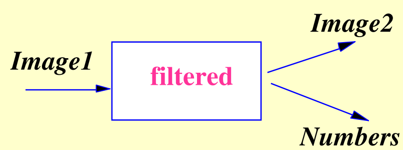


Image Analysis

2 - *Feature Extraction* :

- quality improvment,
- filtering;
- parameters extraction



3 - *Segmentation* :

i.e. partitioning the image
into homogeneous regions
(for some criterion)

Definitions of Mathematical Morphology

Mathematics

Lattice theory for objects or operators in continuous or discrete spaces; Algebraïc,topological and stochastic geometry.

Physics

Signal analysis techniques based on set theory aiming at the study of relations between physical and structural properties.

Signal Processing

Nonlinear signal processing approach based on minimum and maximum operations.

Computer Engineering

Algorithms, software, and hardware tools for developing image processing applications.

Two Basic Structures

Linear signal processing :

The basic structure in linear signal processing is the ***vector space*** *i.e.* a set of ***vectors*** V and a set of ***scalars*** K such that

1) - K is a field ;
   - V is a commutative group

2) - There exists a multiplicative law between scalars and vectors.

An example :

The numerical functions on the plane, or on the space

Two Basic Structures

Mathematical morphology :

The basic structure is a ***complete lattice*** *i.e.* a set $\mathcal{L}$ such that:

1) $\mathcal{L}$ is provided with a ***partial ordering***, *i.e.* a relation $\leq$ with

$$\mathbf{A \leq A}$$
$$\mathbf{A \leq B,\quad B \leq A \quad \Rightarrow \quad A = B}$$
$$\mathbf{A \leq B,\quad B \leq C \quad \Rightarrow \quad A \leq C}$$

2) For each family of elements $\{Xi\} \in \mathcal{L}$, there exists in $\mathcal{L}$:

a greatest lower bound $\wedge\{Xi\}$, called ***infimum*** (or inf.) and a smallest upper bound $\vee\{Xi\}$, called ***supremum*** (or sup.)

Examples :

The subsets of a set; and again the numerical functions.

Basic Operations

Linear Signal Processing

Since the structure is that of a vector space, whose fundamental laws are ***addition*** and ***scalar product***, then

the basic operations are those which preserve these laws, *i.e.* which ***commute*** under them:

$$\Psi(\Sigma \lambda_i f_i) = \Sigma \lambda_i \Psi(f_i)$$

The resulting operator is called ***convolution*.**

Basic Operations

Mathematical Morphology

Since the Lattice structure rests on ***supremum*** and ***infimum***, the basic operations are those which ***preserve*** these fundamental laws, namely

- *ordering Preserving* :

$\{ X \leq Y \Rightarrow \Psi(X) \leq \Psi(Y) \}$ $\Leftrightarrow$ ***increasingness***

- *commuting under supremum* :

$\Psi(\vee X_i) = \vee \Psi(X_i)$ $\Leftrightarrow$ ***Dilation***

- *commuting under infimum* :

$\Psi(\wedge X_i) = \wedge \Psi(X_i)$ $\Leftrightarrow$ ***Erosion***

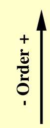
Examples of Lattices

1. Lattices of real or integer numbers:

This ***total*** ordering is given by the succession of the values:

Sup: $\vee$ *Inf*: $\wedge$ (in the numerical sense)

Universal bounds (extreme elements): $-\infty$, $+\infty$

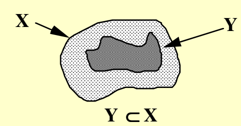


2 Lattice of subsets P(E) of a set E:

The ***partial*** ordering is defined by the inclusion law:

Sup: $\cup$ *Inf*: $\cap$ (set union and intersection)

Universal bouds : E, $\varnothing$



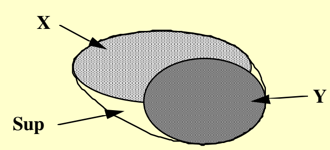
Examples of Lattices

3. Lattice of convex sets:

The order is defined by the inclusion law:

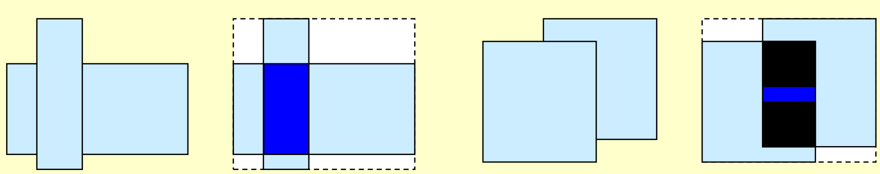
Sup : Convex hull of the union

Inf : $\cap$ (set intersection)



3. Sub-lattice of the rectangles:

In 2-D, the rectangles parallel to the axes form a lattice, but not the squares !



Lattices of Functions

- If E is an arbitrary set, and if T designates $\bar{\mathbf{R}}$, $\bar{\mathbf{Z}}$ one of their closed subsets, then the functions $f : E \to T$ generate in turn a new **lattice**, denoted by T^E, for **the product ordering**

$$f \leq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all} \quad x \in E ,$$

  where sup and inf derive directly from those of T, *i.e.*

$$(\vee f_i)(x) = \vee f_i(x) \qquad (\wedge f_i)(x) = \wedge f_i(x) .$$

  By convention, the same symbol 0 stands for the minimum in T and in T^E.

- In T^E, the **pulse functions** :
  $k_{x,t}(y) = t$ **when** $x = y$; $k_{x,t}(y) = 0$ **when** $x \neq y$
  are **sup-generators,** *i.e.* any $f : E \to T$ is a supremum of pulses.
- The approach extends directly to the products of T type lattices, *i.e.* to ***multivariate functions*** (*e.g. color images, motion*).

Lattices of Partitions

Definition : A **partition** of space E is a mapping D: $E \to \mathcal{P}(E)$ such that

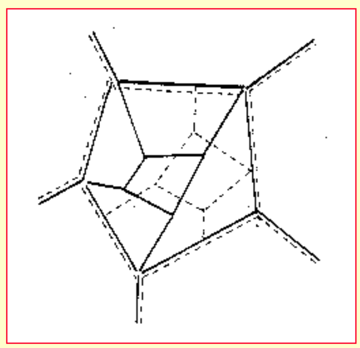
(i) $\forall\ x \in E , \quad x \in D(x)$

(ii) $\forall\ (x, y) \in E,$

either $D(x) = D(y)$

or $D(x) \cap D(y) = \varnothing$

The partitions of E form a **lattice** $\mathcal{D}$ for the ordering according to which $D \leq D'$ when each class of D is included in a class of D'. The largest element of $\mathcal{D}$ is E itself, and the smallest one is the pulverizing of E into all its points.



The sup of the two types of cells is the pentagon where their boundaries coincide. The inf, simpler, is obtained by intersecting the cells.

Atoms , Co-primes and Complement

- A subset L' of lattice L is called a ***sub-lattice*** if it closed under $\vee$ and $\wedge$ and contains the two extremes 0 and m of L.
- A lattice L is ***complemented*** when for every $a \in L$, there exists one $b \in L$ at least such that

$$a \vee b = m \quad ; \quad a \wedge b = 0 .$$

- A non zero element a of a lattice L is an ***atom*** if

$$x \leq a \quad \Rightarrow \quad x = 0 \ \text{ or } \ x = a .$$

- An element $x \in L$ is said to be a ***co-prime*** when

$$x \leq a \vee b \quad \Rightarrow \quad x \leq a \quad \text{or} \quad x \leq b .$$

Sup-generators ; Distributivity

- A lattice L is sup-generated when it has a subset X, called a ***sup-generator***, such that every $a \in L$ is the supremum of the elements of X that it majorates

$$a = \vee \{ x \in X , x \leq a \}$$

When the sup-generators are co-prime (resp. atomic), then lattice L is said to be co-prime (resp. atomic) .

- Lattice L is ***distributive*** if, for all $a , y , z \in L$

$$a \wedge (y \vee z) = (a \wedge y) \vee (a \wedge z) \quad \text{or, equivalently}$$

$$a \vee (y \wedge z) = (a \vee y) \wedge (a \vee z) .$$

- When theses conditions extend to infinity, lattice L is ***infinite distributive***

$$a \wedge (\vee y_i , i \in I) = \vee \{ (a \wedge y_i) , i \in I \}$$

$$a \vee (\wedge y_i , i \in I) = \wedge \{ (a \vee y_i) , i \in I \}$$

(NB : *the two conditions are no longer equivalent* !)

Characterisation of $\mathcal{P}$(E) Lattices

***Theorem** (G.Matheron):* The three following statements are equivalent

- L is ***complemented*** and generated by the class Q of its ***co-primes*** ;
- L is isomorphic to a **$\mathcal{P}$(E)** ***type*** lattice ;
- L is isomorphic to lattice **$\mathcal{P}$(Q)** .

When they are satisfied, L is infinite distributive

Other lattices

- The function lattice T^E is ***infinite distributive*** but not complemented. The pulses are sup-generating ***co-primes***, but they are not atoms .
- The lattice $\mathcal{D}$ of the partitions is ***sup-generated***, but neither distributive nor complemented.

Notion of duality

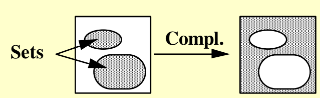
The two laws of Sup. and Inf. play a symmetrical role. Each involution (c) that permutes them generates a duality. More precisely,

Definition: Two operators ψ and ψ* are dual with respect to the involution (c) when:

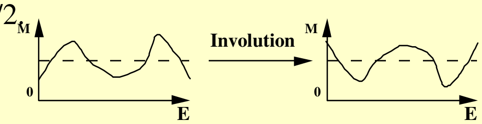
$$\psi (X^c) = [\psi^* (X)]^c$$

Examples of involution **:**

<u>*Lattice of subsets of a set*</u>: The involution is the **complement**. It translates to the classical notion of foreground and background:



<u>*Lattice of real functions bounded by [0,M]*</u>: The involution is the **reflection** with respect to M/2.



Self duality

Linear processing :

- The convolution operation is self dual, that is dual of itself:
  $f * (-g) = - (f * g)$
- This means that positive or negative (bright and dark) components are processed in a symmetrical way.

Mathematical morphology :

- The fundamental duality between Sup. and Inf. translates to all morphological tools.
- In general, morphological operations go by pair and correspond to each other by duality: as examples erosion and dilation, opening and closing.
- However, operators may also be
  - *self-dual*, *i.e.*

$\psi (X^c) = [\psi (X)]^c$ (*e.g.* morph. centre)

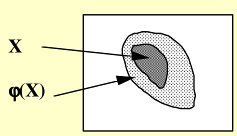
- or *invariant under duality*, *i.e.*

$\psi (X^c) = \psi (X)$ (*e.g.* boundary set in R^n)

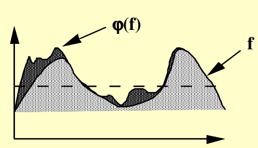
Input Output Comparison

Extensivity anti-extensivity **:** A transformation is extensive if its output is always greater than its input. By duality, it is anti-extensivity when the output is always smaller than the input.

Extensivity : $X \subseteq \Psi (X)$ anti-extensivity : $X \supseteq \Psi (X)$



Set (extensivity)



Function (extensivity)

Idempotence **:** A transformation is idempotent if its output is invariant with respect to the transformation itself

Idempotence: $\Psi [\Psi (X)] = \Psi (X)$

Lattices of Operators

With every lattice $\mathcal{L}$ is associated the class $\mathcal{L}'$ of the operations α: $\mathcal{L} \rightarrow \mathcal{L}$. Now, $\mathcal{L}'$ turns out to be a lattice where :

$$\alpha \leq \beta \quad (\textit{in } \mathcal{L}') \qquad \Leftrightarrow \qquad \alpha(A) \leq \beta(A) \quad \textit{for all } A \in \mathcal{L}$$

$$(\vee \alpha_i)(A) = \vee \alpha_i(A) \qquad (\wedge \alpha_i)(A) = \wedge \alpha_i(A)$$

$$(\textit{ in } \mathcal{L}') \quad (\textit{ in } \mathcal{L}) \qquad (\textit{ in } \mathcal{L}') \quad (\textit{ in } \mathcal{L})$$

- for example, The mappings which are :
  - ***increasing*** **,** - or ***extensive*** **,** - or ***anti-extensive*** ,

  over $\mathcal{L}$ are each a sub-lattice of $\mathcal{L}'$;
- More generally, we shall meet lattices for :
  - ***openings*** - ***filters*** - ***activity*** *etc...*

Part I : Bases

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Adjunction erosion/dilation

- ***Set Erosion*** **:** Operation ε_B commutes under $\cap$ **:**

$$\boldsymbol{\varepsilon_B(\cap X_i)} = \{z: B(z) \subseteq \cap X_i\} = \cap \{z: B(z) \subseteq X_i\} = \cap\, \boldsymbol{\varepsilon_B(X_i)},$$

Therefore, it is effectively an **erosion.**

- ***Adjunction*** **:** The equivalences

$$X \subseteq \varepsilon_B(Y) \quad \Leftrightarrow \quad \{x \in X \Rightarrow B(x) \subseteq Y\} \quad \Leftrightarrow \quad \cup\{B(x), x \in X\} \subseteq Y$$

yield the operation

$$\boldsymbol{\delta_B(X) = \cup\{B(x), x \in X\}}$$

which commutes under $\cup$. The later is thus a **dilation**, said to be **adjoint** of ε. Adjunction is an involution, since by taking the inverse way, we see that ε is adjoint of δ.

- ***Structuring Element*** **:** Since $\delta_B(X) = \cup\{\delta_B(x), x \in X\}$, the mapping ***" structuring element "*** $\boldsymbol{x \rightarrow \delta_B(x) = B(x)}$ suffices to characterise both
  - dilation $\delta : X \rightarrow \delta(X)$ - and erosion $\varepsilon : X \rightarrow \varepsilon(X)$.

Adjunction (II)

The Adjunction Theorem *(E. Gallois....H .Heijmans, Ch. Ronse, J. Serra)***:**
When two operators δ and ε are linked by the equivalence

$$\boldsymbol{X \subseteq \varepsilon(Y) \quad \Leftrightarrow \quad \delta(X) \subseteq Y}$$

then they necessarily form an "***erosion-dilation***" doublet.

- *Proof* : Let be a family Y_i, $i \in I$, and X such that

$$\delta(X) \subseteq \cap Y_i \quad \Leftrightarrow \quad \delta(X) \subseteq Y_i \quad \text{for every } i \in I,$$

By adjunction : first inclusion $\Leftrightarrow$ $X \subseteq \varepsilon(\cap Y_i)$

second inclusion $\Leftrightarrow$ $X \subseteq \varepsilon_B(Y_i)$, $i \in I$, $\Leftrightarrow$ $X \subseteq \cap \varepsilon(Y_i)$

This implies $\varepsilon(\cap Y_i) = \cap \varepsilon(Y_i)$, *i.e.* that ε is an erosion (*id.* for the dilation).

First Representation *(J. Serra)* **:** For any pair (δ,ε) we have :

$$\varepsilon(Y) = \cup\{X : \delta(X) \subseteq Y\} \qquad \delta(X) = \cap\{Y : \varepsilon(Y) \subseteq X\}$$

Curiously, erosion appears here as a union and dilation as an intersection

N.B. *the approach extends to mappings from one lattice into another.*

Representations and Semi-groups

- ***Second Representation Theorem*** *(J.Serra)* **:** Every **increasing** mapping ψ on $\mathcal{P}(E)$ can be written as a union of erosions as follows

$$\psi = \cup \{ \varepsilon_B , B \in \mathcal{P}(E) \} ,$$

with $\varepsilon_B(X) = \psi(B)$ if $X \supseteq B$, and $\varepsilon_B(X) = \varnothing$ otherwise (dual result for the dilation).

This representation generalises G. Matheron's one, for the translation invariant case (II, 14), and extends itself to the complete lattice case.

- ***Semi-groups*****:** The composition product of two dilations (resp. erosions) is still a dilation (resp. erosion). Indeed we have

$$\delta_{B2}\delta_{B1}(X) = \cup \{ B_2(y) , y \in \cup \{ B_1(x) , x \in X \} = \cup \{ \delta_{B2}[B_1(x)] , x \in X \}$$

hence $\delta_{B2}\delta_{B1} = \delta_A$; $\varepsilon_{B2}\varepsilon_{B1} = \varepsilon_A$ **with** $A = \delta_{B2}(B_1)$

[*Semi-group* $\Rightarrow$ *no inverse* $\Leftrightarrow$ ***loss*** *of information.*]

Translation Invariance

- Suppose set E equipped with a translation τ. The translation invariant operations ψ: $\mathcal{P}(E) \rightarrow \mathcal{P}(E)$ are called **τ-mappings** .
- Then, the two basic dilations on $\mathcal{P}(E)$ are
  - the **Minkowski Addition** , which is the **unique** τ-dilation,
  - the **Geodesic Dilation**, which is limited to a given mask
- For all $X \subseteq E$, introduce the **translate** X_b of X according to vector b :

$$X_b = \{x+b, x \in X\}$$

- Moreover we always suppose B to be **symmetrical**, i.e.

$$x \in X \quad \Leftrightarrow \quad -x \in X$$

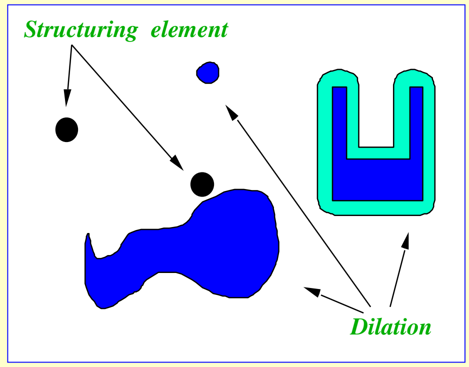
Set Dilation and Minkowski Addition

• The τ-dilations are called Minkowski Additions.

• Each of them is characterized by the transform B of the origin, or Structuring Element.

•By putting $\delta_B(X) = X \oplus B$, we have

$$\boxed{\begin{aligned} X\oplus B &= \cup \{ B_x ,\ x \in X\} \\ &= \cup \{ x + b,\ x\in X,\ b\in B \} \\ &= \cup \{ X_b,\ b\in B \} = B \oplus X \end{aligned}}$$



• As B is symmetrical, $X\oplus B$ is the locus of the centres of B's that hit X:

$$X\oplus B = \{ z:\ B_z \cap X \neq \varnothing\}$$

Set Erosion and Minkowski Subtraction

- The Minkowski subtraction of X by B is, by definition, the erosion $X \ominus B$ *adjoint* to $X \oplus B$.
- *Geometrical interpretation*

$X \ominus B$ turns out to be the locus of the centres z of B_z when the latter is included in X :

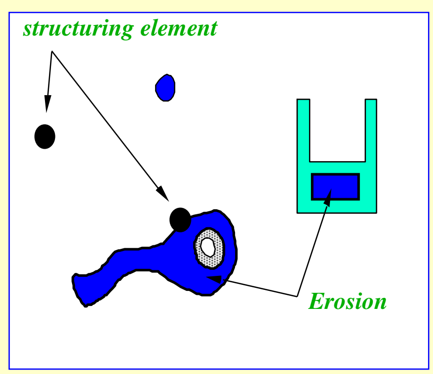
$$\varepsilon_B(X) = X\ominus B = \{ z :\ B_z \subset X \ \}$$

- *$\cap$ - Representation*

$B_z \subseteq X \Leftrightarrow \forall\, b\in B:\ b+z\in X$

$\Leftrightarrow \forall\, b\in B:\ z\in X_{-b}$, hence

$$X\ominus B = \cap \{ X_b, b \in B \}$$



The two Dualities

- ***Adjunction***, already seen, is the duality

$$\mathbf{X \subseteq Y \ominus B \quad \Leftrightarrow \quad X \oplus B \subseteq Y \qquad X, Y \in E.}$$

It characterises the pairs "erosion-dilation". The adjoint term looks like an inverse. In particular, when X, Y and B are **convex and similar**, then

$$X = Y \ominus B \;\Leftrightarrow\; X \oplus B = Y.$$

- Another duality is obtained by taking the ***complement*** *i.e.* in case of an erosion, by putting :

$$\psi(X) = (X^c \ominus B)^c$$

Now, $(X^c \ominus B)^c = [\cap \{(X_b)^c, b \in B\}]^c = \cup \{X_b, b \in B\}$

i.e. $\quad \boldsymbol{\psi(X) = (X^c \ominus B)^c = X \oplus B}$.

The operation dual, under complement, of Minkowski subtraction by B is Minkowski addition by B.

Algebraic Properties of Minkowski Operations

Distributivity

We have the following ***equalities***

$X \oplus (B \cup B') = (X \oplus B) \cup (X \oplus B')$

$X \ominus (B \cup B') = (X \ominus B) \cap (X \ominus B')$

$(X \cap Z) \ominus B = (X \ominus B) \cap (Z \ominus B)$

but only the ***inclusions***

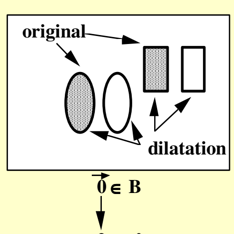
$X \oplus (B \cap B') \subseteq (X \oplus B) \cap (X \oplus B')$

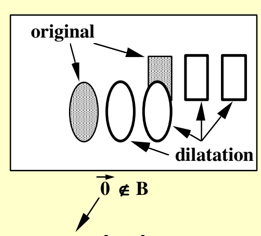
$X \ominus (B \cap B') \supseteq (X \ominus B) \cup (X \ominus B')$

$(X \cup Z) \ominus B \supseteq (X \ominus B) \cap (Z \ominus B)$

Extensivity

$$O \in B \;\Rightarrow\; \begin{cases} \mathbf{X \subseteq (X \oplus B)} \\ \mathbf{(X \ominus B) \subseteq X} \end{cases}$$





Dilation is extensive and erosion anti-extensive iff **B contains the origin**

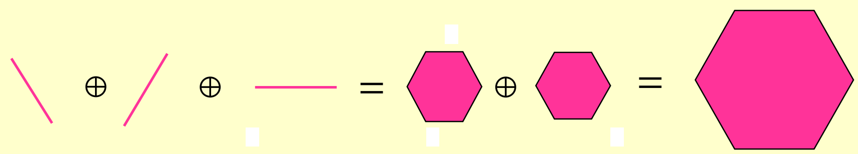
Minkowski Addition by Convex sets

- In the Euclidean space R^n denote by λB the set similar of B by factor λ. Then the semi-goup law:

$$[(X \oplus \lambda B) \oplus \mu B)] = X \oplus (\lambda + \mu) B$$

is satisfied if and only if B is **compact convex** ($x,y \in B \Rightarrow [x,y] \in B$). Moreover, if B is plane and symmetrical, it is equal to a product of dilations by **segments**.

- Practically, the dilation (*resp*. the erosion) of a set X by the convex structuring element λB reduces to λ dilations (*resp*. erosions) by the structuring element B. Iteration acts as a magnification factor.



Edge Effects

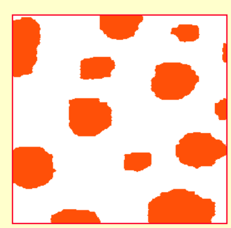
Most of the scenes under study are restrictions, to a rectangle Z, of a larger set X.

- Experimentally, one can access only $X \cap Z$, or $X \cup Z^c$, according to the value 0 ou 1 that one decide to give to the outside. For B symmetrical we have

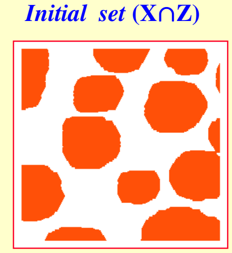
$$(X \cap Z) \ominus B = (X \ominus B) \cap (Z \ominus B) \quad \text{and}$$

$$(X \oplus B) \cap (Z \ominus B) = [(X \cup Z^c) \oplus B] \cap (Z \ominus B).$$

- In other words, the transforms $(X \oplus B)$ and $(X \ominus B)$ are correctly known inside ***mask Z eroded*** itself by B. Worse, when we concatenate a sequence of transformations we soon reduce the mask to $\varnothing$!



Initial set **$(X \cap Z)$**



Dilate **$(X \oplus B) \cap (Z \ominus B)$**

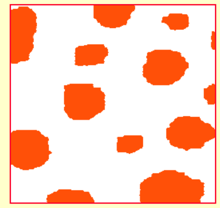
Standard Dilation et Erosion

- To solve the problem, we will reduce progressively the structuring element when it comes near the edge. We (progressively...) loose translation invariance, but the result is provided in the ***whole mask*** Z .
- In such a "**standard**" approach, the working space de définition becomes Z and the structuring element $x \rightarrow B(x) \cap Z$. Dilation and adjoint erosion are written as follows:

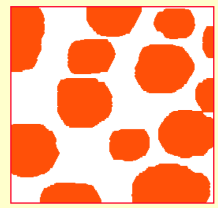
$$\delta_B(X) = \cup\{ B(x) \cap Z \,;\; x \in X \cap Z \}$$
$$\varepsilon_B(X) = \{ x: B_x \cap Z \subseteq X \cap Z \,;\, x \in X \cap Z \}$$

Similarly, the duality under complement ***in Z*** is

$$\psi^*(X) = Z \setminus \psi\,(Z \setminus X).$$



Initial set **(X∩Z)**

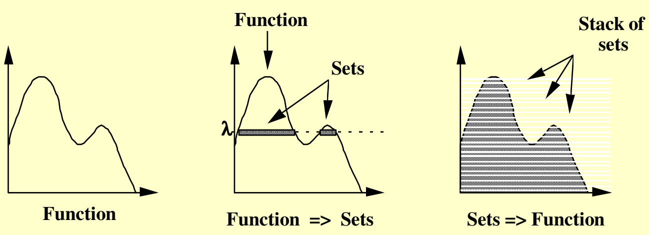


Standard Dilation of **(X∩Z)**

Equivalence between Sets and Functions

A function can be viewed as a **stack of decreasing sets**. Each set is the intersection between the **umbra** of the function and a horizontal plane.

$$X_\lambda\,(f) = \{ x \in E \,,\, f(x) \geq \lambda \} \quad \Leftrightarrow \quad f(x) = \sup\,\{\lambda : x \in X_\lambda\,(f)\,\} \qquad (*)$$



It is equivalent to say that f is upper semi-continuous or that the X_λ's are closed. Conversely, given a family $\{X_\lambda\}$ of closed sets such that

$$\lambda \geq \mu \;\Rightarrow X_\lambda \subseteq X_\mu \quad \text{and} \quad X_\lambda = \cap\,\{X_\mu \,,\, \mu < \lambda\,\}$$

there exists a ***unique*** u.s.c. function f whose sections are the X_λ's.

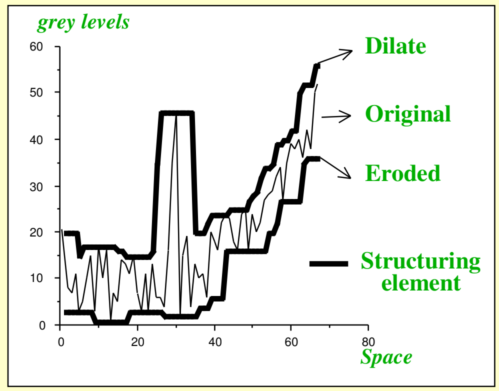
Dilation and erosion by a flat structuring element

Definition **:** The dilation (erosion) of a function by a flat structuring element B is introduced as the dilation (erosion) of each set $X_f(\lambda)$ by B.They are said to be **planar** .

This definition leads to the following formulae :

$$(f\oplus B)(x) = \sup\{ f(x-y),\ y\in B \}$$

$$(f\ominus B)(x) = \inf \{ f(x-y),\ -y\in B \}$$



- Erosion shrinks positive peaks. Peaks thinner that the structuring element disappear. As well, it expands the valleys and the sinks.
- Dilation produces the dual effects.

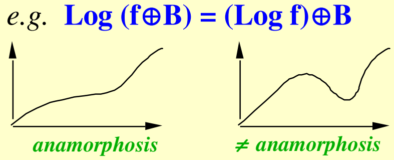
Properties of the planar operators

- Erosion and dilation, with flat or non flat structuring elements, have basically the same properties as those stated for sets.
- In addition, the use of ***flat*** structuring elements provides the three following specific advantages :

Commute under anamorphosis

An anamorphosis is an increasing continuous mapping of the grey level values.

e.g. **Log (f⊕B) = (Log f)⊕B**



Stability

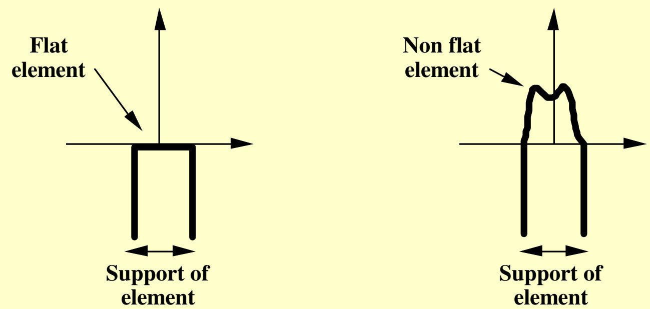
The class of the functions which take *n* given values is preserved (any n-bit image is tranformed into an n-bit image).

Implementation

A transformation based on flat structuring elements can be implemented either level by level, or numerically.

Non Planar Structuring Elements

- Planar structuring elements can be viewed as a function of constant level, equals to 0, and whose support is the structuring set. These structuring elements can be generalised by introducing weights. The resulting elements, no longer planar, are also called « ***non flat*** ».



Dilation of Functions by non flat Elements

Definition

Dilation and erosion of function f by the (non flat) function h are given by the relations

$$(f \oplus h)(x) = \sup_{y \in E} [f(x-y) + h(y)]$$

$$(f \ominus h)(x) = \inf_{y \in E} [f(x+y) - h(y)]$$

Remark:

Since the images under study traduce physical phenomena, one shall take care to provide f and f with consistent units.

Comparison with Convolution

We can establish a parallelism between the formulae of dilation and of erosion and that of convolution .

Sum	<=>	Sup or Inf
Product	<=>	Sum

convolution :

$$(h*f)(x) = \sum_{y \in E} f(x-y) \,.\, h(y)$$

dilation:

$$(f \oplus h)(x) = \sup_{y \in E} [f(x-y) + h(y)]$$

Morphological Gradients

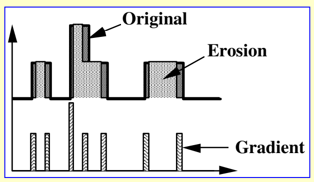
The goal of gradients transformations is to highlight contours. In digital morphology, three **Beucher's gradients** based on **the unit disc** are defined:

Gradient by erosion :

- It is the residue between the **identity** and an **erosion** , *i.e.*:

  for sets $g^{-}(X) = X / (X \ominus B)$

  for functions $g^{-}(f) = f - (f \ominus B)$

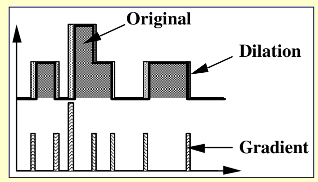


Gradient by dilation :

- It is the residue between a **dilation** and the **identity**, *i.e.* :

  for sets $g^{+}(X) = (X \oplus B) / X$

  for functions $g^{+}(f) = (f \oplus B) - f$



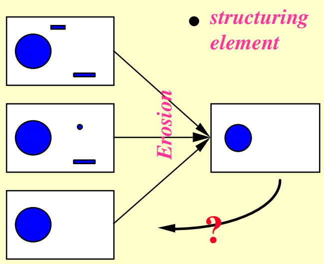
Part I : Bases

- ordering and lattices
- erosion and dilation
- opening and closing

Adjunction Opening and Closing

The problem of an inverse operator

Several different sets may admit a same erosion, or a same dilate. But among all possible inverses, there exists always a smaller one (a larger one). It is obtained by composing erosion with the adjoint dilation (or *vice versa*) .



The mapping is called ***adjunction opening***, and is denoted by

$\gamma_B = \delta_B \, \varepsilon_B$ (*general case)*

$XoB = [(X \ominus B) \oplus B]$ (τ-*operators)*

By commuting the factors δ_B and ε_B we obtain the ***adjunction closing***

$\varphi_B = \varepsilon_B \, \delta_B$ *(general case)*,

$X \bullet B = [X \oplus B) \ominus B]$ (τ-*operators).*

Properties of Adjunction Opening and Closing

Increasingness

Adjunction opening and closing are increasing as products of increasing operations.

(Anti-)extensivity

By doing $Y = \delta_B(X)$, and then $X = \varepsilon_B(Y)$ in adjunction $\delta_B(X) \subseteq Y \Leftrightarrow X \subseteq \varepsilon_B(Y)$, we see that:

$\boxed{\delta_B \, \varepsilon_B (X) \subseteq X \subseteq \varepsilon_B \, \delta_B (X)}$ hence $\varepsilon_B \, (\delta_B \, \varepsilon_B) \subseteq \varepsilon_B \subseteq (\varepsilon_B \, \delta_B)\varepsilon_B \Rightarrow$ $\boxed{\varepsilon_B \, \delta_B \, \varepsilon_B = \varepsilon_B}$

Idempotence

The erosion of the opening equals the erosion of the set itself. This results in the idempotence of γ_B and of φ_B :

$\varepsilon_B \, (\delta_B \, \varepsilon_B) = \varepsilon_B \Rightarrow \delta_B \, \varepsilon_B \, (\delta_B \, \varepsilon_B) = \delta_B \, \varepsilon_B$ *i.e.* $\gamma_B \, \gamma_B = \gamma_B$

Finally, if $\varepsilon_B(Y) = \varepsilon_B(X)$, then $\gamma_B(X) = \delta_B \, \varepsilon_B(X) = \delta_B \, \varepsilon_B(Y) \subseteq Y$. Hence, γ_B is the smallest inverse of erosion ε_B .

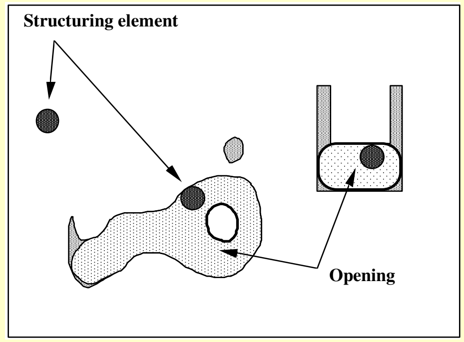
Amending Effects of Adjunction Opening

Geometrical interpretation

$z \in \gamma_B(X) \Leftrightarrow z \in B_y$ and $y \in X \ominus B$

hence $\boxed{z \in \gamma(X) \Leftrightarrow z \in B_y \subseteq X}$

- the opened set $\gamma_B(X)$ is the union of the structuring elements B(x) which are included in set X.
- In case of a τ–opening, $\gamma_B(X)$ is the zone swept by the structuring element when it is constrained to be included in the set.



When B is a disc, the opening amends the caps, removes the small islands and opens isthmuses.

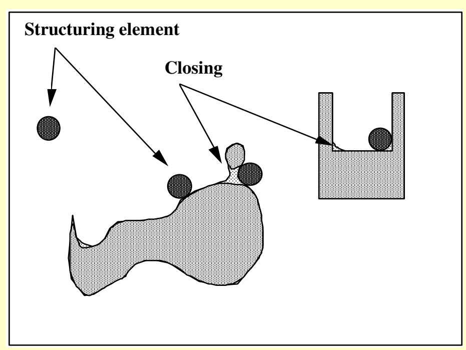
Amending Effects of Adjunction Closing

Geometrical interpretation

- By taking the complement in the definition of XoB we see that

$$X \bullet B = [(X \oplus \check{B}) \ominus \check{B}]$$

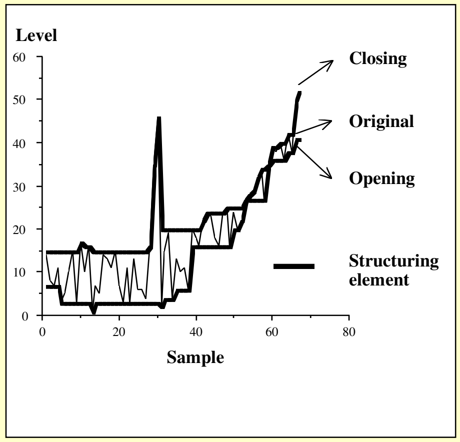
- The τ-closing is the complement of the domain swept by B as it misses set X. Note that in most of the practical cases, set $\check{B}$ is symmetrical, *i.e.* identical to B.
- Note that a shift of the origin affects both erosions and dilations, but does not act on openings and closings.



When B is a disc, the closing closes the channels, fills completely the small lakes, and partly the gulfs.

Effects on Functions

- The adjunction opening and closing create a ***simpler function*** than the original. They smooth in a nonlinear way.
- The ***opening*** (closing) removes ***positive*** (negative) ***peaks*** that are thinner than the structuring element.
- The opening (closing) remains below (above) the original function.



Algebraic Opening and Closing

The three basic properties of adjunction openings $\delta\varepsilon$ and closings $\varepsilon\delta$ are also the **axioms** for the algebraic notion of an opening and a closing.

Definition **:** In algebra, any transformation which is:

- increasing, anti-extensive and idempotent is an (algebraic) opening,
- increasing, extensive and idempotent is called a (algebraic) closing.

Particular cases **:**

Here are two very easy ways for creating algebraic openings and closing:

1) Compute various opening (closing) and take their supremum (or the infimum in case of closings).
2) Use a ***reconstruction*** process

Invariant Elements

Let $\mathcal{B}$ be the image of lattice L under the algebraic opening γ, *i.e.* $\mathcal{B} = \gamma(L)$. Since γ is idempotent, set $\mathcal{B}$ generates the family of ***invariant sets*** of γ :

$$\mathbf{b \in \mathcal{B} \Leftrightarrow \gamma(b) = \mathcal{B}}.$$

1/ Classe $\mathcal{B}$ is ***closed under sup***. For any family $\{b_j, j \in J\} \subseteq \mathcal{B}$, we have

$$\gamma(\vee b_j, j \in J) \geq \vee\{\gamma(b_j), j \in J\} = \vee(b_j, j \in J)$$

by increasingness, and the inverse inequality by anti-extensivity de γ. Moreover , $0 \in \mathcal{B}$. Note that γ does not commute under supremum.

2/ Therefore, γ is the ***smallest extension*** to L of the identity on $\mathcal{B}$, *i.e.*

$$\boldsymbol{\gamma(x) = \vee\{b : b \in \mathcal{B}, b \leq x\}, \qquad x \in L \qquad (1)}.$$

[The right member is an invariant set of γ smaller than x, but also that contains $\gamma(x)$.]

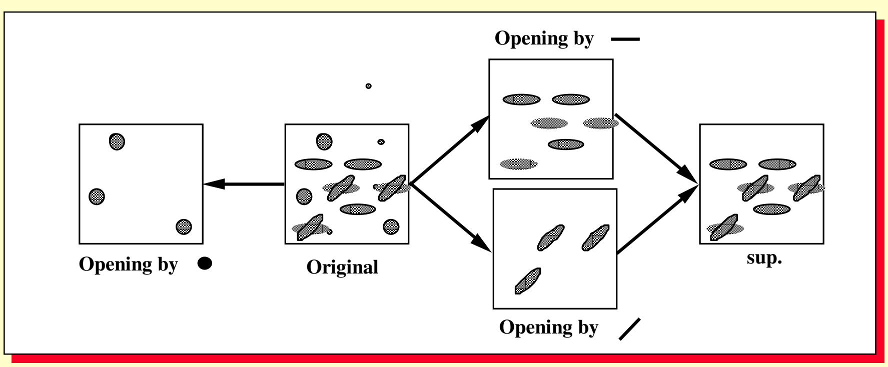
Suprema of Openings

***Theorem* :**

- Any supremum of openings is still an opening.
- Any infimum of closings is still a closing.

***Application* :**

For creating openings with specific selection properties, one can use structuring elements with various shapes and take their supremum.



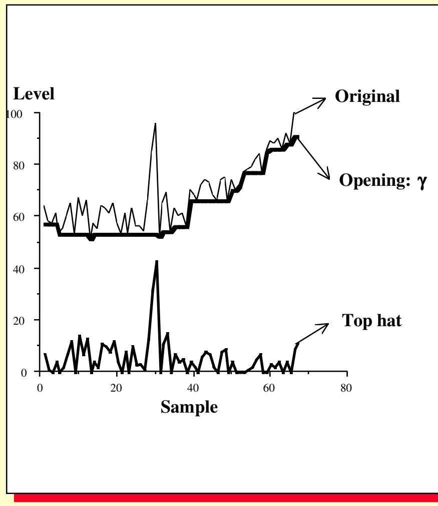
Top-hat (opening residue)

Sets

- The top-hat extracts the objects that have not been eliminated by the opening. That is, it removes objects larger than the structuring element.

Functions

- The top-hat is used to extract contrasted components with respect to the background. The basic top-hat extracts positive components and the dual top hat the negative ones.
- Typically, top-hats remove the slow trends, and thus performs a contrast enhancement.



Numerical top-hat

- White top-hat : $f - \gamma f$ (γ is an opening)
- Black top-hat : $\phi f - f$ (ϕ is a closing)

White top-hat
(SE 3 × 3)

Original image
(256 × 256 pixels)

Black top-hat
(SE 3 × 3)

Granulometry: an intuitive approach

- *Granulometry* is the study of the size characteristics of sets and of functions. In physics, granulometries are generally based on **sieves** ψ_λ of increasing meshes $\lambda > 0$. Now,
  - by applying sieve λ to set X, we obtain the over-sieve $\psi_\lambda(X) \subseteq X$;
  - if Y is another set containing X, the Y-over-sieve, for every λ, is larger than the X-over-sieve, *i.e.* $X \subseteq Y \Rightarrow \psi_\lambda(X) \subseteq \psi_\lambda(Y)$;
  - if we compare two different meshes λ and μ such that $\lambda \geq \mu$, the μ-over-sieve is larger than the λ–over-sieve, *i.e.* $\lambda \geq \mu \Rightarrow \psi_\lambda(X) \subseteq \psi_\mu(X)$
  - finally, by applying the largest mesh λ to the μ–over-sieve, we obtain again the λ–over-sieve itself, *i.e.* $\psi_\lambda \psi_\mu (X) = \psi_\mu \psi_\lambda (X) = \psi_\lambda (X)$
- Such a description of the physical sieving suggests to resort to **openings** for an adequate formalism. The sizes of the structuring elements will play the role of the the sieves meshes.

Granulometry: a formal approach

- *Matheron Axiomatics* defines a granulometry as a family $\{\gamma_\lambda\}$

  i) of openings depending on a positive parameter λ,

  ii) and which decrease as λ increases: $\lambda \geq \mu > 0 \;\Rightarrow\; \gamma_\lambda \leq \gamma_\mu$.
- This second axiom is equivalent to a **semi-group** where the composition of two operations is equal to the stronger one, namely

$$\gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda = \gamma_{\sup(\lambda,\mu)} \qquad (1)$$

- If $\mathcal{B}_\lambda$ et $\mathcal{B}_\mu$ stand for the invariant elements of γ_λ and of γ_μ respectively, then we easily see that

$$(1) \quad \Leftrightarrow \quad \mathcal{B}_\lambda \subseteq \mathcal{B}_\mu$$

- When γ_λ's are **adjunction openings**, *i.e.* $\gamma_\lambda(X)=X \circ \lambda B$ (with ***similar*** structuring elements), then the granulometry axioms are fulfilled if and only if B is compact and ***convex***

Granulometry: a Formal Approach

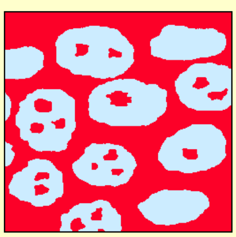
$[\{ \lambda \geq \mu > 0 \;\Rightarrow\; \gamma_\lambda \leq \gamma_\mu \} \quad \Rightarrow \quad \gamma_\lambda = \gamma_\lambda \gamma_\lambda \leq (\gamma_\lambda \gamma_\mu \vee \gamma_\mu \gamma_\lambda) \leq \gamma_\lambda$;
conversely, $\gamma_\lambda = \gamma_\mu \gamma_\lambda$ et $\gamma_\lambda \leq I \;\Rightarrow\; \gamma_\lambda \leq \gamma_\mu$ hence semi-group (1).]

- If $\mathcal{B}_\lambda$ et $\mathcal{B}_\mu$ stand for the invariant elements of γ_λ and of γ_μ respectively, then we easily see that

$$(1) \quad \Leftrightarrow \quad \mathcal{B}_\lambda \subseteq \mathcal{B}_\mu$$

- When γ_λ's are **adjunction openings**, *i.e.* $\gamma_\lambda(X) = X \circ \lambda B$ (with ***similar*** structuring elements), then the granulometry axioms are fulfilled if and only if B is compact and ***convex***.

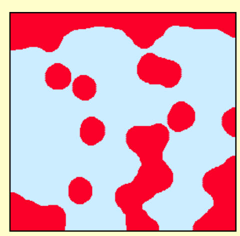
An Example of Granulometry

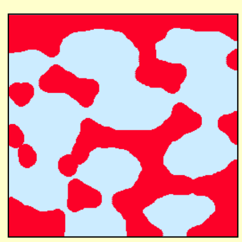


By duality, the families of closings $\{\varphi_\lambda, \lambda > 0\}$ increasing in λ generate ***anti-granulometries*** of law

$$\varphi_\lambda \varphi_\mu = \varphi_\mu \varphi_\lambda = \varphi_{\sup(\lambda,\mu)} .$$

Here, from left to right, closings by increasing discs.



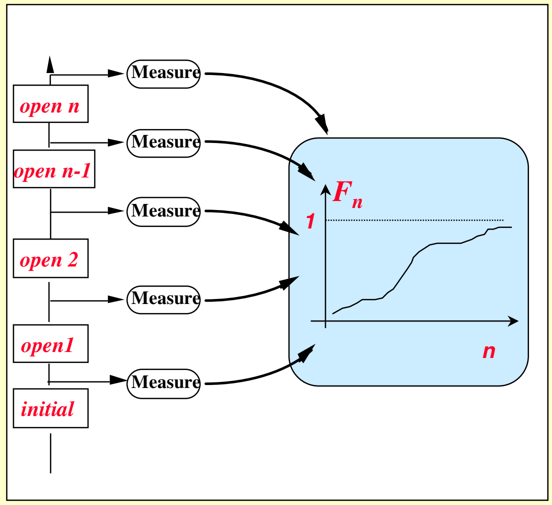


Granulometry and Measurements

- A granulometry is computed from a pyramid of openings, or closings, whose each element is given a size, λ say;
- Value λ is the similarity ratio holding on the involved structuring element(s) .
- At the output of each filter, the area is measured (set case), or the integral in case of functions, M_λ say. Then, the monotonic curve

$$F_\lambda = 1 - M_\lambda / M_0$$

is a ***Distribution Function*** .



Granulometric Spectrum

One also uses the *granulometric spectrum*, that is the derivative of the granulometric distribution function.