Quiz 1

(1) Is $d(x,y) = \frac{|x-y|}{1+|x-y|}$ a metric on $\mathbb{R}$?

Solution: Clearly $d(x,y) \geq 0$ and $d(x,x) = 0$. If $d(x,y) = 0$, then $|x-y| = 0$ $\Rightarrow x = y$. Now for $x, y, z \in \mathbb{R}$, then $|x-z| \leq |x-y| + |y-z|$ and so $|x-z| \leq |x-y|(1+|y-z|) + |y-z|(1+|x-y|) + |x-y||y-z||x-z|$. So,

$$\begin{array}{ll} & |x-z|(1+|x-y|)(1+|y-z|) \\ \leq & |x-y|(1+|y-z|) + |y-z|(1+|x-y|) + |x-y||y-z||x-z| \\ & +|x-z|[|y-z| + |x-y|(1+|y-z|)] \\ \leq & |x-y|(1+|y-z|) + |y-z|(1+|x-y|) \\ & +|x-z|[|x-y|(1+|y-z|) + |y-z|(1+|x-y|)] \end{array}.$$

This implies that

$$d(x,z) \leq d(x,y) + d(y,z).$$

(2) Let $a \neq b$ be two points in a metric space X. Prove that there is a $\delta > 0$ such that $N_\delta(a) \cap N_\delta(b) = \emptyset$.

Solution: Let $r = d(a,b) > 0$. Take $\delta = r/2$. Then for $x \in N_\delta(a)$, $d(x,b) \geq d(a,b) - d(a,x) > r/2 = \delta$. Thus, $N_\delta(a) \cap N_\delta(b) = \emptyset$.

(3) Let (X,d) be a metric space and $Y \subset X$. Define ρ by $\rho(a,b) = d(a,b)$ for all $a, b \in Y$. If U is an open set in the metric space (Y,ρ), then show that there is an open set V in (X,d) such that $U = V \cap Y$.

Solution: Since U is an open subset of Y, for each $x \in U$ there is a $\delta_x > 0$ such that $N_{\delta_x}(x,Y) \subset U$. So, $U = \cup_{x\in U} N_{\delta_x}(x,Y)$. Let $V = \cup_{x\in U} N_{\delta_x}(x,X)$. Then V is an open set in X and $V \cap Y = \cup_{x\in U}(N_{\delta_x}(x,X) \cap Y) = \cup_{x\in U} N_{\delta_x}(x,Y) = U$.

(4) Let E be a subset of a metric space X. Show that $E^0 = \overline{E^C}^C$.

Solution: $x \in E^0 \Leftrightarrow N_\delta(x) \subset E$ for some $\delta > 0 \Leftrightarrow N_\delta(x) \cap E^c = \emptyset$ for some $\delta > 0 \Leftrightarrow x \notin \overline{E^C}$.

(5) Determine all open subsets and compact subsets in a discrete metric space.

Solution: Since $N_1(x) = \{x\}$, all subsets are open. Let K be any compact subset. Then since $K = \cup_{x\in K}\{x\}$ and $\{x\}$ are all open, there are $x_1, \cdots, x_k \in K$ such that $K = \{x_1, \cdots, x_k\}$. Thus, K is a finite set. Since finite subsets are compact, we get that finite subsets are the only compact sets.