Mid-Semester Exam Analysis 1 Time 3:00hrs Total Marks 40
Answer any five questions, each question is worth 8marks

1. Let (a_n) and (b_n) be sequences converging to a and b respectively. Prove that
   (i) $a_n + b_n \to a + b$ and $ra_n \to ra$ for any $r \in \mathbb{R}$,
   (ii) (a_n) is a bounded sequence.

2. Let (x_n) and (y_n) be bounded sequences. Prove that $\underline{\lim}(-x_n) = -\overline{\lim}x_n$ and

$$\begin{aligned}\underline{\lim}x_n + \underline{\lim}y_n \le \underline{\lim}(x_n + y_n) &\le \overline{\lim}x_n + \underline{\lim}y_n \\ &\le \overline{\lim}(x_n + y_n) \le \overline{\lim}x_n + \overline{\lim}y_n.\end{aligned}$$

Solution: It is easy to see that $-\sup_{x\in E} x = \inf_{x\in E} -x$, we have $\inf_{k\ge n} -x_n = -\sup_{k\ge n} x_n$, hence

$$\liminf -x_n = \sup_n(-\sup_{k\ge n} x_n) = -\inf_n \sup_{k\ge n} x_n = -\limsup x_n.$$

We first prove $\limsup x_n + \liminf y_n \le \limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$. Now $\sup_{k\ge n} x_k + y_k \le \sup_{k\ge n} x_k + \sup_{k\ge n} x_k$, hence letting $n \to \infty$, we get that

$$\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n.$$

This also implies that

$$\limsup x_n \le \limsup(x_n + y_n) + \limsup(-y_n) = \limsup(x_n + y_n) - \liminf y_n$$

hence

$$\limsup x_n + \liminf y_n \le \limsup(x_n + y_n).$$

Now we also have

$$\liminf -x_n + \limsup -y_n \le \limsup -(x_n + y_n) \le \limsup -x_n + \limsup -y_n$$

hence

$$-\limsup x_n - \liminf y_n \le -\liminf(x_n + y_n) \le -\liminf x_n - \liminf y_n$$

hence

$$\liminf x_n + \liminf y_n \le \liminf(x_n + y_n) \le \limsup x_n + \liminf y_n.$$

3. (i) If (a_n) is a sequence and r is a limit point of (a_n), then show that there is a subsequence (a_{k_n}) of (a_n) such that $a_{k_n} \to r$.

   (ii) If (a_n) is defined by

$$a_1 = 0, \quad a_{2m} = \frac{a_{2m-1}}{2} \quad a_{2m+1} = \frac{1}{2} + a_{2m}$$

   for any $m \geq 1$. Find $\liminf a_n$ and $\limsup a_n$.

   Solution: $a_2 = 0$, $a_3 = 1/2$, $a_4 = 1/2^2$, $a_5 = \frac{1}{2} + \frac{1}{2^2}$. We first claim that $a_{2m} = \sum_{k=2}^{m} \frac{1}{2^k}$ and $a_{2m+1} = \sum_{k=1}^{m} \frac{1}{2^k}$ for all $m \geq 2$. Claim is true for $m = 2$. Suppose for some $m \geq 2$, $a_{2m} = \sum_{k=2}^{m} \frac{1}{2^k}$ and $a_{2m+1} = \sum_{k=1}^{m} \frac{1}{2^k}$. Then $a_{2m+2} = \frac{a_{2m+1}}{2} = \sum_{k=2}^{m+1} \frac{1}{2^k}$ and $a_{2m+3} = \frac{1}{2} + a_{2m+2} = \frac{1}{2} + \sum_{k=2}^{m+1} \frac{1}{2^k} = \sum_{k=1}^{m+1} \frac{1}{2^k}$. Thus by induction the claim is true for any $m \geq 2$.

   This proves that $a_{2m} = (1 - \frac{1}{2^{m+1}}) - \frac{1}{2}$ and $a_{2m+1} = (1 - \frac{1}{2^{m+1}})$. This proves that $|a_m| \leq 1$ and $a_{2m} < a_{2m+1}$ for all m.

   For any $n \geq 1$, $\sup_{k \geq n} a_k = \sup_{2k+1 \geq n} a_{2k+1}$ and hence $\limsup a_n = \limsup a_{2m+1} = \lim a_{2m+1} = 1$ and similarly we can show that $\liminf a_n = \frac{1}{2}$

4. (i) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} (n-1)^{\frac{1}{n}} = 1$.

   Solution: Put $x_n = (n-1)^{\frac{1}{n}} - 1$ and $y_n = n^{\frac{1}{n}} - 1$. Then $\frac{n(n-1)}{2} x_n^2 \leq n - 1$ and $\frac{n(n-1)}{2} y_n^2 \leq n$, hence $x_n \leq \sqrt{\frac{2}{n}}$ and $y_n \leq \sqrt{\frac{2}{n-1}}$. Thus, $x_n \to 0$ and $y_n \to 0$.

   (ii) Prove that every Cauchy sequence converges.

5. (i) If $|a_n| \leq c_n$ for all n and $\sum c_n$ converges, prove that $\sum a_n$ also converges.

   (ii) Let (a_i) be a decreasing sequence of non-negative numbers. Then prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^k a_{2^k}$ converges.

6. (i) If the sequence of partial sums of $\sum a_n$ is bounded and (b_n) is a decreasing or increasing sequence converging to zero, prove that $\sum a_n b_n$ converges.

   (ii) If $\sum a_n$ converges and (b_n) is a bounded monotonic sequence, prove that $\sum a_n b_n$ converges.

   Solution: (i) is proved in the class if (b_n) is decreasing. Suppose (b_n) is increasing. We know that $b_n \to \sup b_n = 0$. Hence $b_n \leq b_{n+1} \leq 0$. So, $-b_n \geq -b_{n+1}$ and $-b_n \to 0$. By Theorem proved in the class, $\sum a_n(-b_n)$ converges, hence $\sum -a_n(-b_n) = \sum a_n b_n$ also converges. For (ii). Since $\sum a_n$ converges, sequence of partial sums of $\sum a_n$ is bounded. Since (b_n) is monotonic and bounded, $b_n \to b \in \mathbb{R}$. Now $(b_n - b)$ is monotonic and $b_n - b \to 0$. By (i), $\sum a_n(b_n - b)$ converges. By basic properties, we have $\sum b a_n$ also converges. This proves that $\sum a_n b_n = \sum (a_n(b_n - b) + b a_n)$ also converges.

7. (i) Find the radius of convergence of the following series

$$\frac{1}{3}+\frac{1}{5}z+\frac{1}{3^2}z^2+\frac{1}{5^2}z^3+\frac{1}{5^3}z^4+\frac{1}{5^3}z^5+\cdots$$

Solution: $a_n=\frac{1}{3^{n/2+1}}$ if n is even and $a_n=\frac{1}{5^{\frac{n+1}{2}}}$ if n is odd.

For $n\geq 5$,

$$\sup_{k\geq n}|a_k|^{\frac{1}{k}}=\begin{cases} a_n^{\frac{1}{n}} & n \text{ is even} \\ a_{n+1}^{\frac{1}{n+1}} & n \text{ is odd}\end{cases}$$

since when n (≥ 5) is odd,

$$5^{\frac{n+1}{2}}<3^{\frac{n+1}{2}+1}.$$

This implies that $\limsup a_n^{\frac{1}{n}}=\frac{1}{\sqrt{3}}$. Thus, the radius of convergence is $\sqrt{3}$.

(ii) Let (a_n) be a sequence. Define $p_n=|a_n|+a_n$ and $q_n=|a_n|-a_n$. Prove that

(a) $\sum p_n$ and $\sum q_n$ converge if and only if $\sum a_n$ converges absolutely;

(b) if $\sum a_n$ and $\sum p_n$ converge, then $\sum a_n$ converges absolutely.

Solution: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges. Hence $\sum |a_n|\pm a_n$ converges. This implies that $\sum p_n$ and $\sum q_n$ converge.

suppose $\sum p_n$ and $\sum q_n$ converge. Then $\sum \frac{p_n+q_n}{2}=\sum |a_n|$ converges. Thus, $\sum a_n$ converges absolutely.

If $\sum a_n$ and $\sum p_n$ converge, then $\sum p_n-a_n=\sum|a_n|$ converges, hence $\sum a_n$ converge absolutely.