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Technical note

Estimation of number–area–frequency dimensions of surface water bodies

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Abstract. This paper proposes a technique that includes a set of mathematical morphological transformations to estimate the frequency dimension. The dimension computed through a power law relationship is tallied with the dimension computed through a correlational plot. This technique is demonstrated on a two-dimensional section embodying a large number of surface water bodies, extracted from remotely sensed data, situated randomly, and the frequency dimension (D) for surface water bodies yields straight-line dependence of $\ln C(r)$ (correlational integral) on $\ln(r)$ (radius of structuring template). The correlational integral is computed for two aspects by considering the number of water bodies and their corresponding occupied areas. The number–frequency dimension and the area–frequency dimension computed through correlational plots yield straight-line dependencies with slopes that are greater than unity but less than 2.0 (1.3 and 1.7, respectively).

1. Introduction

A section of surface water bodies is one of the best examples of a natural fractal occurring in any landscape. Several properties such as possession of non-integral dimensionality and self-similarity characterize the fractals. Several methods are available to compute the dimensions of such fractals. Recent studies on surface water bodies applying mathematical techniques include automatic computation of dimensional parameters (Sagar *et al.* 1995a), distribution of surface water bodies according to their shapes and sizes (Sagar *et al.* 1995b), ranking of lakes according to the dynamical behaviour in the time domain (Sagar and Rao 1995 a, b, c), fractal-based dimensional analysis (Sagar 1994, Sagar and Rao 1995 d), morphological dynamical behaviour of lakes (Sagar *et al.* 1998) and fractal and morphometric relationship of the topological network of water bodies (Sagar *et al.* 1999). In all these studies, the application of mathematical morphology, fractals and non-linear concepts are shown on surface water bodies extracted from remotely sensed data. In the present paper,

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one of these mathematical techniques is used to compute the frequency dimension, which can be taken as quantifying parameters in terms of analytical values to characterize the fractals that come as 'dust'—isolated points, thinly sprinkled over some range of space. The idea for this study was based on the technique of Grassberger and Procaccia (1983) to compute the correlation dimension of time series data of chaotic signals, which is one of the most widely used dimensions and is more accessible experimentally, in particular to compute the dimension of strange attractors which more often possess the dust-like objects of various lengths (e.g. Henon attractor). The proposed technique to compute the frequency dimension is based on mathematical morphological transformations. It is demonstrated on a section of water bodies, being one of the best examples of dust-like sprinkled objects in nature. This dimension, which is less than the geometric fractal dimension and information dimension, can be computed according to a power law:

$$C(r) \sim r^{D} \tag{1}$$

where $C(r)$ is the correlational integral, r is the radius and D is the correlation dimension.

2. Experimental

A discrete binary image (M) that contained water bodies, defined as a finite subset of IR^2, was used. The geometrical properties of M, which contained water bodies (set) and non-water bodies (set compliment), were subjected to morphological functionals by means of a defined sub-image (or kernel) that is here termed a structuring template (S). A bounded S (figure 1) that possesses a designed shape that is thought of as a probe of M was used. Constraints that correspond to the four principles of the theory of mathematical morphology (Serra and Luc Vincent 1982), such as invariance under translation, compatibility with change of scale, local knowledge and the upper semi-continuity, are important on morphological transformations—*erosion* to shrink, *dilation* to expand and cascade processes performed by means of structuring templates that are represented by a compact subset of Euclidean space. M and S are sets of Euclidean space with elements m and s, respectively; $m = (m_1 \ldots, m_n)$ and $s = (s_1 \ldots, s_n)$ being n-tuple elements, morphological set transformations can be performed on M by means of S. Dilation and erosion combines and subtracts, respectively, two sets using vector addition and subtraction of set elements, one coming from M and the other from S. The dilation (equation 2) and erosion (equation 3) of M with S are defined as the set of all points 'm' that the translated S_m intersects and contains in M:

$$M \oplus S = \{m\colon S_m \cap M\} = \cup_{s \in S} M_s \tag{2}$$

$$M \ominus S = \{m\colon S_m \subseteq M\} = \cap_{s \in S} M_s \tag{3}$$

where $S = \{s\colon s \in S\}$, i.e. S rotated 180° around the origin and $S = \hat{S}$ in the present context; hence, dilation and erosion are akin to Minkowski addition and subtraction

$$\begin{array}{ccccc} & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & \end{array}$$

Figure 1. Circle type of structuring template.

(Maragos and Schafer 1986). S with its characteristic information is shown schematically in figure 1.

The size of S can be increased as

$$S \oplus S \oplus S \oplus \ldots \oplus S = S_n \tag{4}$$

According to equation (4), multiscale dilation and erosion is written as

$$\left.\begin{array}{l}(M \oplus S) \oplus S \oplus \ldots \oplus S = M \oplus S_n \\ (M \ominus S) \ominus S \ominus \ldots \ominus S = M \ominus S_n\end{array}\right\} \tag{5}$$

3. A sample study

These basic morphological set transformations were used to compute the frequency dimension of a set M, which, in discrete form, contained a large number (1718) of randomly situated surface water bodies of various sizes extracted from IRS-1C remotely sensed data of a region between the geographical coordinates 18°00′ and 18°30′ N and 83°15′ and 83°45′ E. Morphological set transformations were performed systematically to distribute these surface water bodies and to compute the number according to their sizes. This distribution procedure was based on the size of S. During the cascade of erosion–dilation, the water bodies that are smaller than the size of S vanish by leaving the bigger water bodies. The number and the corresponding areas of vanished water bodies are denoted, respectively, as $N\{M/(M \ominus S_n) \oplus S_n\}$ or $N\{M/(M o S_n)\}$ and $A\{M/(M \mathrm{o} S_n)\}$, where a solidus denotes set difference. The number and areal extents of retained water bodies after cascade of erosion–dilation were used to define the number–area–correlational integrals (Grassberger and Procaccia 1983) or number–area–distribution functions (Delfiner 1972).

To determine the number–frequency dimension (D_1), the number of water bodies that have a smaller (Euclidean) diameter than some given diameter ($2r$) of S were counted. As $2r(S_n)$ varies, so does the relative count N; then, the correlational integral ($C(r)$) (Grassberger and Procaccia 1983, Shroeder 1991) was defined as the total count divided by the squared number of water bodies. From the notion of $C(r)$ introduced by Grassberger and Procaccia (1983, Shroeder 1991), it was redefined (equation 6) in morphological terms as the ratio between the number of water bodies vanished after the cascade of erosion–dilation by means of S of radius $r(S_n)$ and the total number of surface water bodies of all sizes. Similarly, by computing the occupied area of corresponding water bodies, the area–frequency dimension (D_2) was computed by following equation (8). $C(r)$ (Grassberger and Procaccia 1983) was introduced a decade earlier with the terms number–area–distribution functions by Delfiner (1972).

$$\text{number–}C[r(S_n)] = \frac{N\{M/(M \mathrm{~o~} S_n)\}}{[N(M)]^2} \tag{6}$$

$$\text{area–}C[r(S_n)] = \frac{A\{M/(M \mathrm{~o~} S_n)\}}{[A(M)]^2} \tag{7}$$

where $\{M/(M \mathrm{o} S_n)\}$ is the set difference between the original set, containing water bodies of several sizes, and the filtered water bodies after cascade of erosion–dilation transformation. The radius of the bounded S (figure 1) that traverses the origin is denoted as $r(S_n)$, and D_1 and D_2 are then defined by the initial slope:

$$D = \frac{\ln C[r(S_n)]}{\ln r(S_n)} \qquad r(S_n) > r(S_{n-1}) > \ldots > r(S) \tag{8}$$

4. Results and conclusions

A set theory based transformation, cascade of erosion–dilation, was adopted here to compute the frequency dimension. This transformation filters the water bodies that are smaller than the size of the structuring template, given in table 1. The computed number–area–correlational integrals are also shown in table 1. In figure 2, the logarithm of these correlational integrals $[\ln C(r)]$ are plotted as functions of the logarithm of a radius of structuring template $[\ln(r)]$, which according to the power law relation (equation 1) should yield straight lines of positive slopes. The slopes of these lines are the frequency dimensions D_1 and D_2. Figure 2 shows the experimental determination of D_1 and D_2 for the randomly situated surface water bodies which yield straight-line dependence of $\ln N\{M/(M \mathrm{o} S_n)\}/[N(M)]^2$ and $A\{M/(M \mathrm{o} S_n)\}/[A(M)]^2$ on $\ln r(S_n)$ with slopes $D_1 = 1.3069$ and $D_2 = 1.7$. It is deduced that the computed frequency dimension through the correlation integral introduced by Grassberger and Procaccia (1983) is exactly similar to the number and area distribution functions defined by Delfiner (1972) in the context of grain size analysis. It is expected that this new technique that demonstrates the computation of frequency dimension of the randomly situated objects of various sizes and shapes (e.g. water bodies, river basins, channel networks, islands, hills and certain geomorphic features) will emerge strongly as more attention is focused on the usage of morphological set transformations in the contexts of physical and geophysical problems. In any case, the estimation of frequency dimension is quite simple and can be automatically performed on the data extracted from remotely sensed digital data.

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Table 1. Distributed surface water bodies and correlational integrals.

Diameter of structuring template in pixel units	Number of vanished surface water bodies	Areas of vanished water bodies in pixel units	Number–correlation integral	Area–correlational integral
5	386	17 019	0.00013078	0.000001718011
7	787	21 299	0.000266642	0.000002150091
9	1059	37 041	0.000358798	0.000003739165
11	1262	67 328	0.000427575	0.000006796537
13	1420	76 583	0.000481107	0.000007730798
17	1718	99 530	0.000582072	0.000010047221

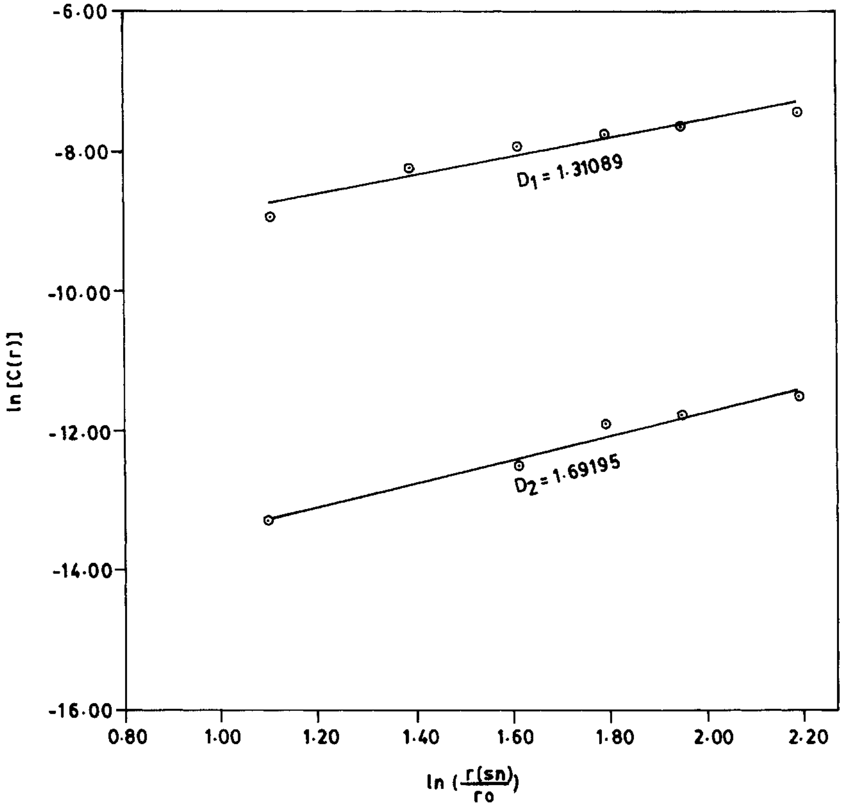

Figure 2. Determination of frequency dimensions from the number–area–correlation integral.

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