

Recall

$\left\{ \begin{array}{l} \text{for } S - \text{sample space was countable} \\ \text{Discrete } \mathcal{F} - \mathcal{P}(S) \text{ all subsets of } S \\ \mathcal{P} - \text{probability } \mathcal{F} \rightarrow [0,1] \end{array} \right.$

$S = [0,1]$ $\mathcal{P}$ - "probability" on S

$$\Rightarrow \mathbb{P}(\{x\}) = 0 \quad \forall x \in S$$

New Definition of σ -algebra

$$\mathcal{F} \subseteq \mathcal{P}(S) \quad \left\{ \begin{array}{l} \textcircled{1} S \in \mathcal{F} \\ \textcircled{2} A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \\ \textcircled{3} A_1, A_2, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \end{array} \right.$$

$$S = \mathbb{R}$$

S - sample space $\mathcal{F}$ - σ -field on S $P: \mathcal{F} \rightarrow [0,1]$

$$(1) P(S) = 1$$

$$(2) \{E_n\}_{n=1}^{\infty} E_n \in \mathcal{F} \quad E_n \cap E_m = \emptyset \quad m \neq n$$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

how to choose a number "uniformly" in $(0,1)$?

- $A \subseteq [0,1]$ for which length cannot be defined

$S = \mathbb{R}$ $\mathcal{B} \equiv \sigma$ -field on $S = \mathbb{R}$ that was smallest σ -field containing all intervals Borel σ -field

$$\mathcal{I} = \{\mathcal{F} \mid \mathcal{F} \text{ - } \sigma\text{-field, } \mathcal{F} \text{ contains intervals}\} \quad \mathcal{B} = \bigcap_{\mathcal{F} \in \mathcal{I}} \mathcal{F}$$

$$S = \mathbb{R}$$

$\mathcal{F} =$ Borel σ -fields

standard Event space

$[a,b)$ $[a,b]$, (a,b) , $(a,b]$

or unions, complements

of above, $-\infty \leq a < b \leq \infty$

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \rightarrow f: \mathbb{R} \rightarrow \mathbb{R}$$

$$P(A) = \int_A f(x) dx$$

$$\mathbb{P}((0,1)) = 1 = |(0,1)|$$

$$A = (a,b) \subseteq (0,1)$$

$$\mathbb{P}(A) = \int_A f(x) dx = \int_a^b 1 dx = b-a = \frac{|A|}{|(0,1)|}$$

$$A = \left[\frac{2}{5}, \frac{3}{5}\right) \cup \left[\frac{4}{5}, 1\right]$$

$$\mathbb{P}(A) = \int_{\frac{2}{5}}^{\frac{3}{5}} 1 dx + \int_{\frac{4}{5}}^1 1 dx = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = \frac{|A|}{|(0,1)|}$$

$$B \subseteq (0,1)^c$$

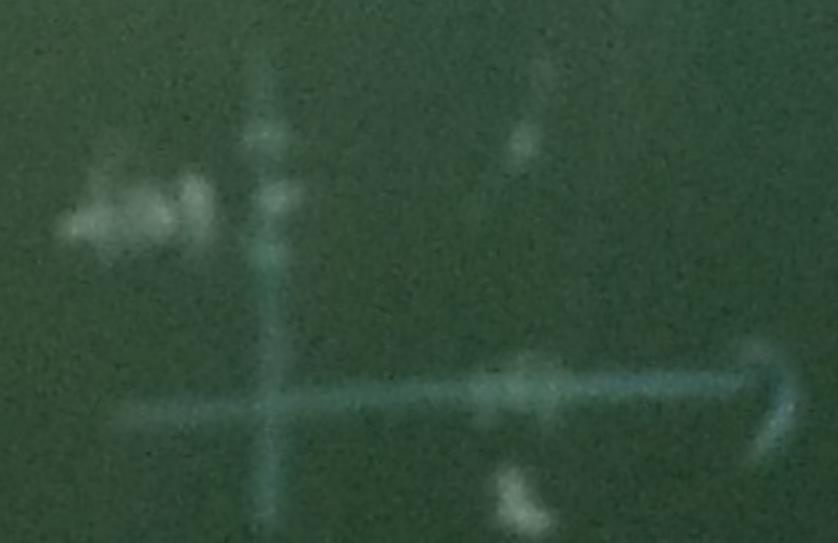
$$\mathbb{P}(B) = \int_B f(x) dx = \int_B 0 dx = 0$$

$$\rightarrow f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) \geq 0 \quad x \in \mathbb{R}, \quad \int_{-\infty}^{\infty} f(x) dx = \int_0^1 f(x) dx = 1, \quad "f \text{ is piecewise continuous}"$$

$$\mathbb{P}(A) = \int_A f(x) dx$$

Definition: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function at $x \in \mathbb{R}$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ such that
 $|f(x) - f(y)| < \varepsilon$ whenever $y \in (x - \delta, x + \delta)$



f is continuous function on $\mathbb{R}$ if f is continuous at all $x \in \mathbb{R}$

f is piecewise continuous if $\mathcal{P} = \{J_k \mid J_k \neq \emptyset \text{ } J_k \text{ is } J_k\text{-interval } 1 \leq k \leq n\}$
 such that $J_k \cap J_l = \emptyset \quad k \neq l$ and $\bigcup_{k=1}^n J_k \subset \mathbb{R}$
 & f is continuous on each J_k .

$f: \mathbb{R} \rightarrow \mathbb{R}$ is called density function if

(1) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(2) f is piecewise continuous on $\mathbb{R}$

(3) $\int_{-\infty}^{\infty} f(x) dx = 1$

$$P(A) = P(a, \infty) = \int_a^{\infty} f(x) dx = 0$$

Theorem 5.1.4: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a density function

let $S = \mathbb{R}$, $\mathcal{F}$ - events space (Borel sets)

$P: \mathcal{F} \rightarrow [0, 1]$

is a probability on S .

$$P(A) = \int_A f(x) dx$$

f is called
"density" or
"density function"
for
Probability P

Proof

$$P(\mathbb{R}) = \int_{\mathbb{R}} f(x) dx = \int_{-\infty}^{\infty} f(x) dx = 1 \quad (\text{Axiom ①} \checkmark)$$

$$A \in \mathcal{F} \quad 0 \leq P(A) = \int_A f(x) dx \leq 1$$

$$\{E_n\}_{n=1}^{\infty} \quad E_n \cap E_m = \emptyset \quad n \neq m \quad E_n \in \mathcal{F}$$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \int_{\bigcup_{n=1}^{\infty} E_n} f(x) dx \stackrel{\text{Axiom 2}}{=} \sum_{n=1}^{\infty} \int_{E_n} f(x) dx = \sum_{n=1}^{\infty} P(E_n) \quad (\text{Axiom ②} \checkmark)$$

P is a probability on $\mathcal{F}$. $\square$

Example 5.1.5

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^{\infty} f(x) dx$$

$$= \int_{-\infty}^0 0 dx + \int_0^1 3x^2 dx + \int_1^{\infty} 0 dx$$

$$\left(\int_a^b f(x) dx = F(b) - F(a) \right) = 0 + \left. x^3 \right|_0^1 + 0$$

$$= 1 - 0 = 1$$

$$P(A) = \int_A f(x) dx \quad A \in \text{Event space}$$

$$P((0,1)) = 1$$

$$B \subseteq (0,1)^c \quad P(B) = 0$$

$$A = \left[\frac{2}{5}, \frac{3}{5}\right]$$

"Uniform" $P(A) = \frac{1}{5}$

$$P(A) = \int_{2/5}^{3/5} 3x^2 dx = x^3 \Big|_{2/5}^{3/5} = \frac{27-8}{125} = \frac{19}{125} \neq \frac{1}{5} = |A|$$

Intuition

$$f(2/5) = 3\left(\frac{2}{5}\right)^2 = 4\frac{3}{5} = 4f(1/5)$$



"Probability of choosing a number in a very small interval around $2/5$ is 4 times that of choosing around $1/5$ "

Random

Definition

X: S

Random Variable $\int f(x) dx = \int g(x) dx = 1$ (norm 1)

Definition 5.1.6 Let $(S, \mathcal{F}, P)$ be a probability space.

$X: S \rightarrow \mathbb{R}$ is called a random variable if

for B real set $B \subseteq \mathbb{R}$ $X^{-1}(B) \in \mathcal{F}$.

$$\{s \in S \mid X(s) \in B\} =$$

- inverse image of an event in $\mathbb{R}$, under X , is an event in $\mathcal{F}$.

Definition 5.1.7 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a density function. Let X be a random variable on $(\mathcal{S}, \mathcal{F}, P)$ is

said to have a probability density function f if

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Analysis $\rightarrow$

$$P(X \in A) = \int_A f(x) dx$$

Such an X is called an absolutely continuous random variable

Lemma 51.8 Let X be a continuum random variable with probability density function $f: \mathbb{R} \rightarrow \mathbb{R}$. Then

$$P(X=a) = 0 \quad \forall a \in \mathbb{R}$$

Proof

$$P(X=a) = P(a \leq X \leq a) = \int_a^a f(x) dx = 0 \quad \square$$

Recall:

X is a random variable with

p.d.f. [Probability density function]

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

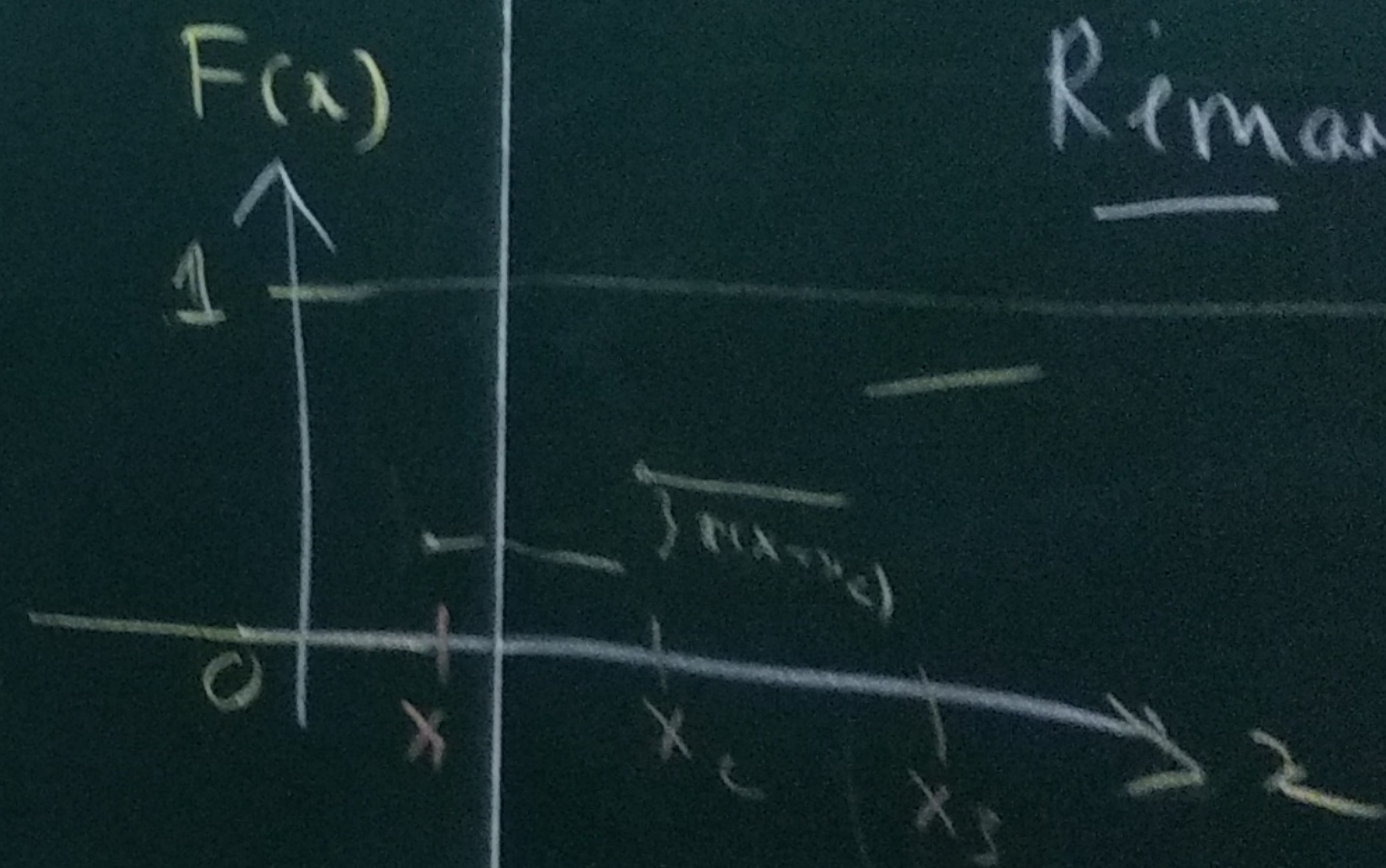
$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

X - continuous random variable

Lemma 51.7 X is continuous $\Rightarrow P(X=a) = 0 \quad \forall a \in \mathbb{R}$

Definition

Remark



Definition 5.2.4

X is a random variable

$$F: \mathbb{R} \rightarrow [0, 1]$$

given by

$$F(x) = P(X \leq x) \quad x \in \mathbb{R}$$

} F

is called the distribution function of X .

Remark.

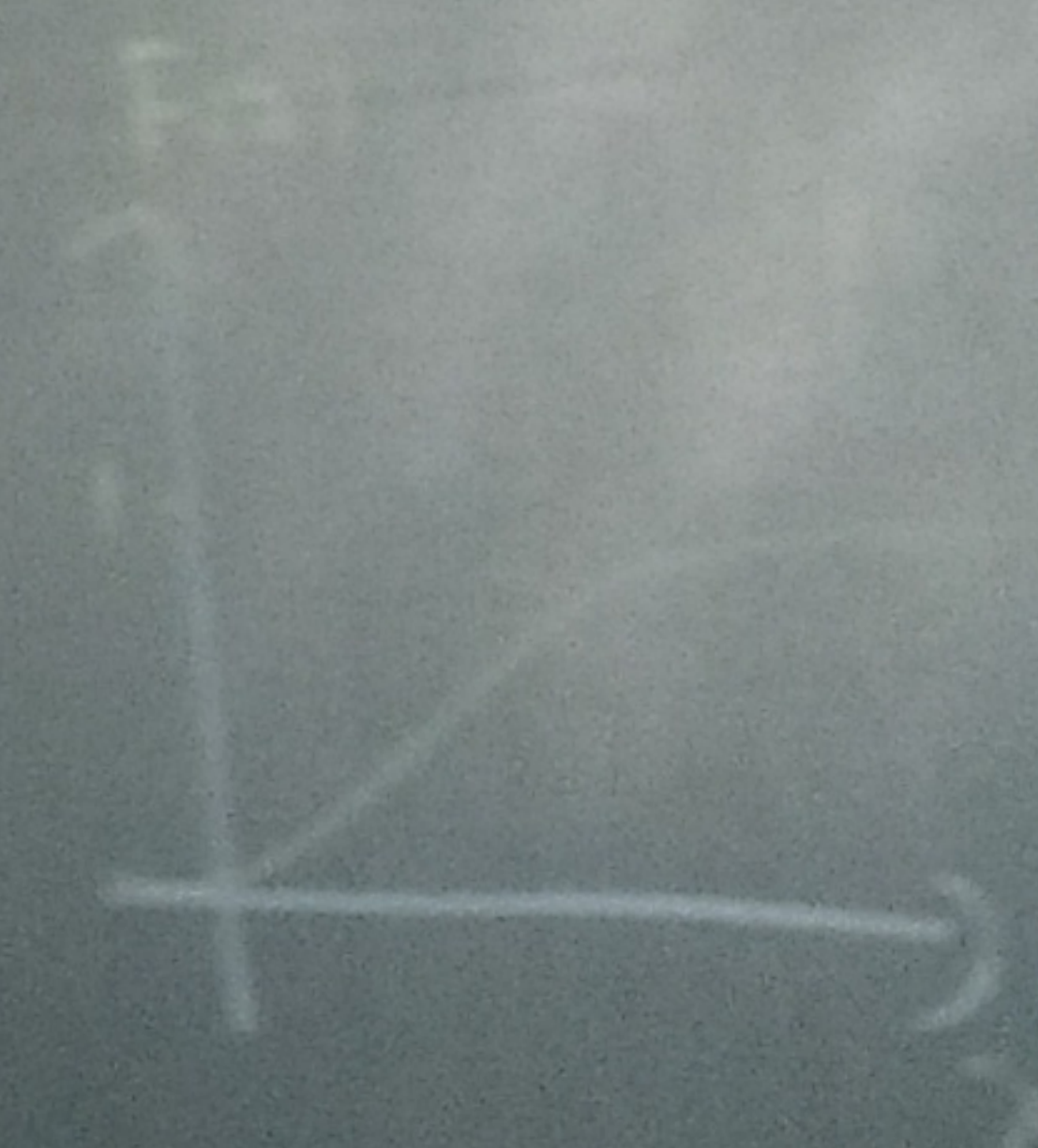
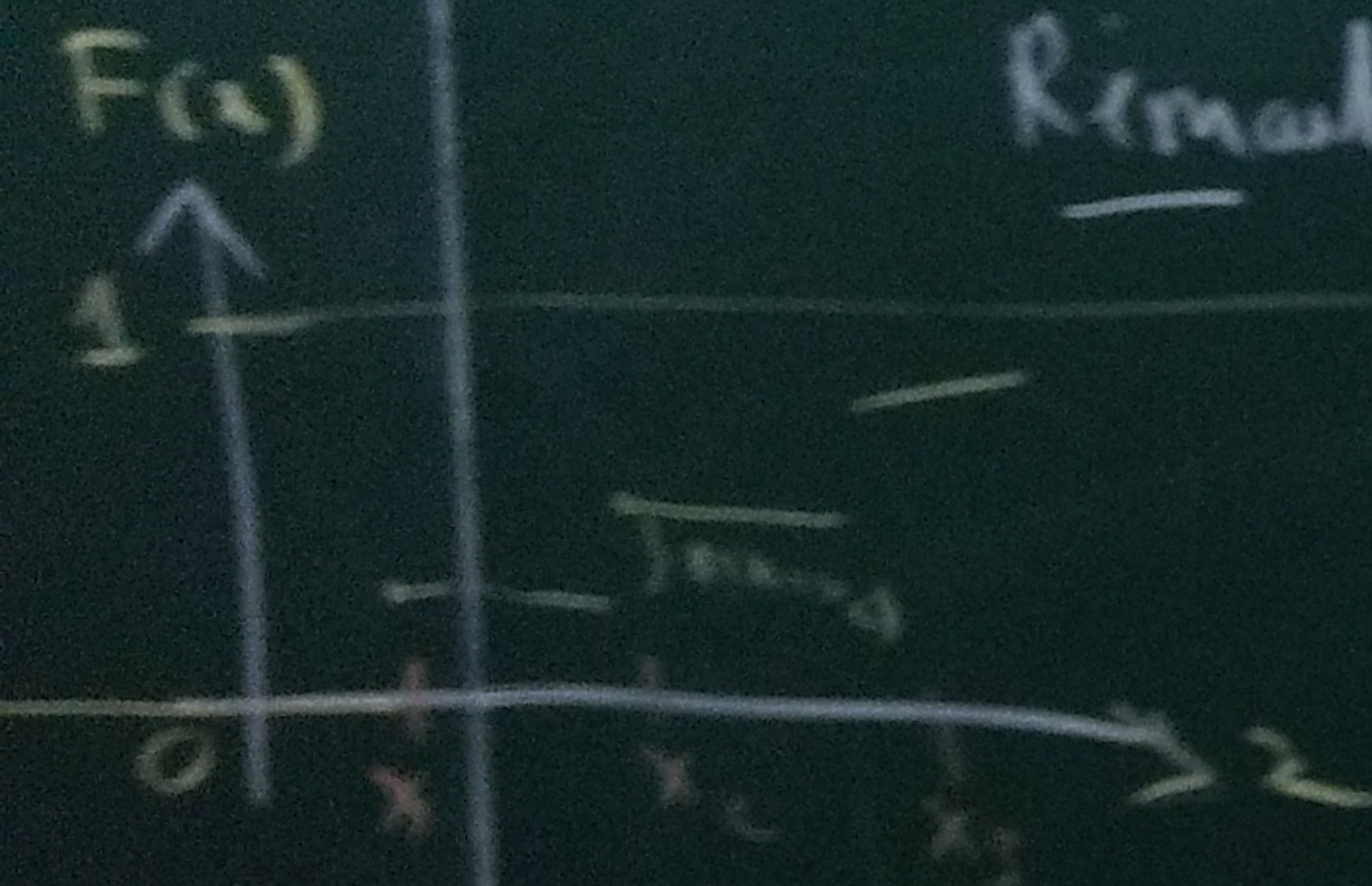
X is discrete random variable $X: S \rightarrow T$ $T \subseteq \mathbb{R}$, countable

$$F(x) = P(X \leq x) = \sum_{x_i \in T, x_i \leq x} P(X = x_i)$$

X is a continuous random variable with pdf $f: \mathbb{R} \rightarrow \mathbb{R}$

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) = \int_{-\infty}^x f(y) dy$$

-(*)



(*) - Con sequence (Analysis 2)

$f: \mathbb{R} \rightarrow \mathbb{R}$ is piecewise
continuous.

$\Rightarrow$

$$F'(x) = f(x)$$

whenever f is continuous at x .

Common Continuous Distributions

$X \sim \text{Uniform}(a, b)$

Let (a, b) be an open interval $-\infty < a < b < \infty$

X is said to Uniform (a, b) random variable if X has a p.d.f given

by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$F: \mathbb{R} \rightarrow [0,1]$$

$$F(x) = \int_{-\infty}^x f(y) dy$$

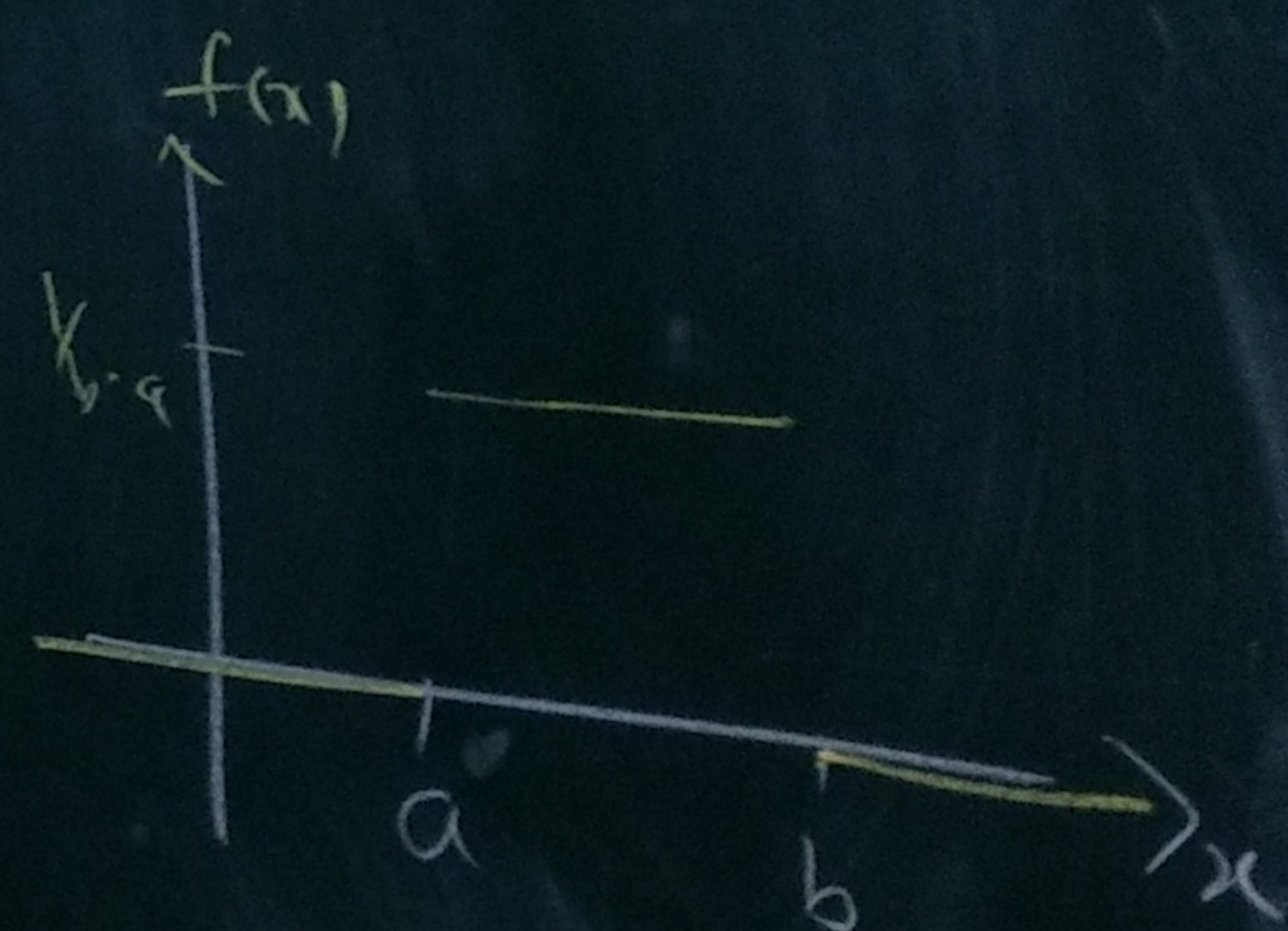
$$= 0$$

$$x \leq a$$

$$F(x) = \int_a^b \frac{1}{b-a} dy = \frac{b-a}{b-a} = 1 \quad x \geq b$$

$$a < x < b, \quad F(x) = \int_{-\infty}^x f(y) dy = \int_{-\infty}^a 0 dy + \int_a^x \frac{1}{b-a} dy$$

$$= \frac{x-a}{b-a}$$



$X \sim \text{Exp}(\lambda)$

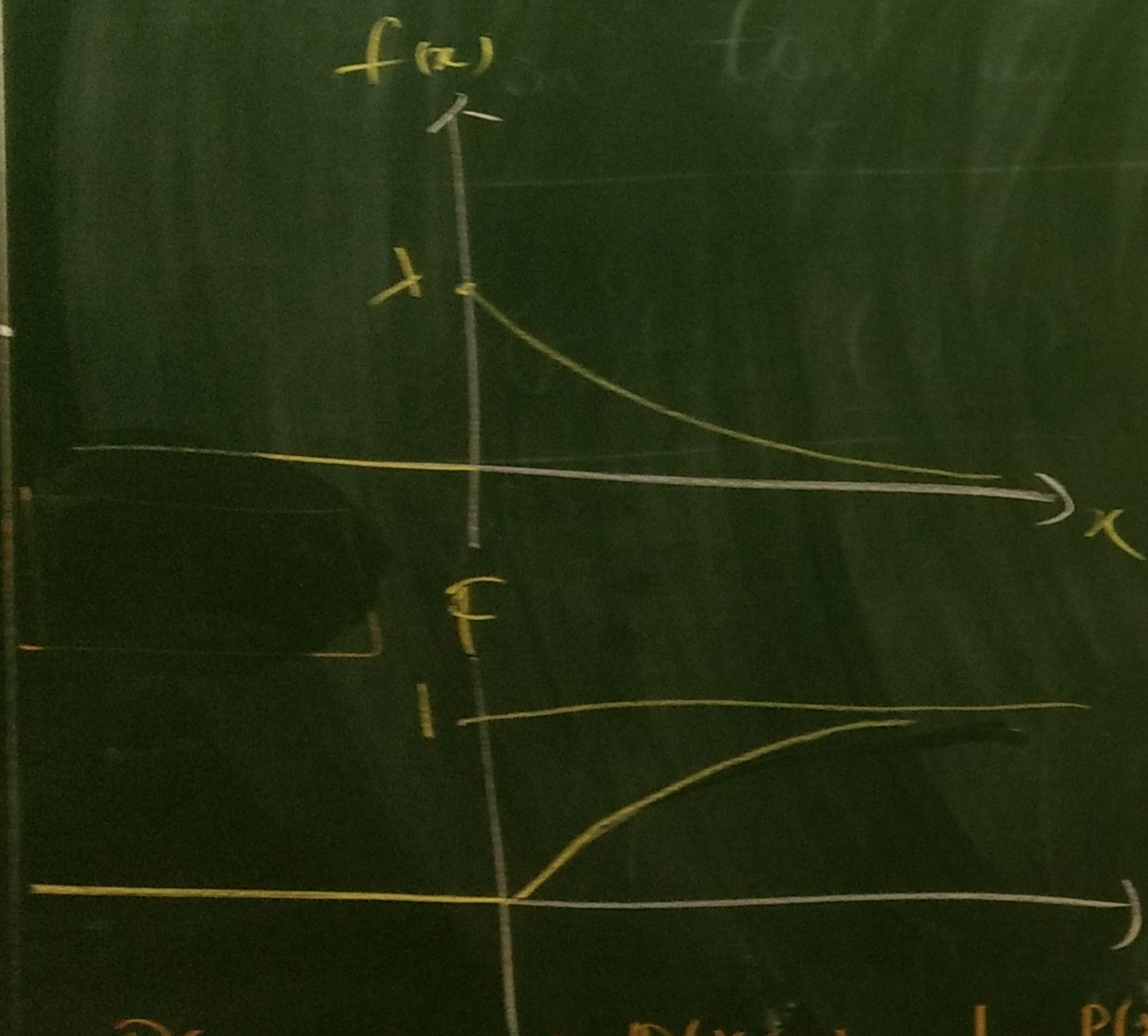
$$\left\{ \begin{array}{l} N(t) \equiv \text{Amount of radioactive material at time } t \\ N(t) \approx N(0) e^{-\lambda t} \\ \text{ie } \frac{N(t)}{N(0)} \approx e^{-\lambda t} \end{array} \right.$$

Wait a random variable $P(X \geq t) = e^{-\lambda t}$

X be a continuous random variable with p.d.f f given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}, \text{ where } \lambda > 0$$

Such an X is called an Exponential (λ) random variable



$$F(x) = \int_{-\infty}^x f(y) dy = \begin{cases} 0 & x \leq 0 \\ \int_0^x \lambda e^{-\lambda y} dy & x > 0 \end{cases}$$

$$\begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0 \end{cases} = \begin{cases} 0 & x \leq 0 \\ -e^{-\lambda y} \Big|_0^x & x > 0 \end{cases}$$

$$P(X > x) = 1 - P(X \leq x) = 1 - P(X \leq x) = 1 - F(x) = e^{-\lambda x}$$

Memoryless Property of Exponential

$\lambda, t > 0$

$$P(X \geq s+t | X \geq t) = \frac{P(X \geq s+t, X \geq t)}{P(X \geq t)}$$

$$X = \text{Exp}(\lambda)$$

Model - waiting
time in Queue

$$= \frac{P(X \geq s+t)}{P(X \geq t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$= P(X \geq s)$$

$X \sim \text{Normal}(\mu, \sigma^2)$: - Let $\mu \in \mathbb{R}$ $\sigma > 0$

X is said to normally distributed with parameters μ and σ^2

if it has a p.d.f $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

$$-\infty < x < \infty$$

known

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

The normal distribution is also referred to as Gaussian distribution.

- It is Central & Extremely important with regard to applications.

$X_1, X_2, \dots, X_n$ are i.i.d Bernoulli(p) $0 < p < 1$

$$S_n = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$$

Poisson approximation

$$p = \frac{\lambda}{n}$$

$$P(S_n = k) \longrightarrow \frac{e^{-\lambda} \lambda^k}{k!} \text{ as } n \rightarrow \infty$$

Normal approximation

De Moivre Central Limit Theorem

$$0 < p < 1 \quad -\infty \leq a < b < \infty$$

$$Y_n = \frac{S_n - np}{\sqrt{np(1-p)}} \quad (\text{standardised } S_n)$$

$$\lim_{n \rightarrow \infty} P(a < Y_n \leq b) =$$

$$\int_a^b \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$$

Normal (0, 1)

$$\frac{e^{-\frac{(x-0)^2}{2 \cdot 1^2}}}{\sqrt{2\pi} \cdot 1}$$