Recall

$X : S \to T_1$ (discrete) $T_i \subseteq \mathbb{R}$, T_i is countable $i=1,2$

$Y : S \to T_2$

Independence: X & Y are independent

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

for all $A \subseteq T_1$, $B \subseteq T_2$

$$\Updownarrow$$

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\,\mathbb{P}(Y = y) \quad \forall\, x \in T_1,\ y \in T_2$$

(Joint Probability) (product of marginals)

• **Joint distribution** X, Y are two discrete random variables

φ : is joint distribution of (X, Y)

$$\underbrace{\varphi(\{x,y\}) = \mathbb{P}(X=x, Y=y)}$$

$x \in \mathrm{Range}(X)$, $y \in \mathrm{Range}(Y)$ (hard to determine from marginals)

Given joint distribution

Marginal distribution of X $\leftarrow$ $\mathbb{P}(X=x) = \sum_{y \in \mathrm{Range}(Y)} \varphi(\{x,y\}) = \sum_{y \in \mathrm{Range}(Y)} \mathbb{P}(X=x, Y=y)$

$$\mathbb{P}(Y=y) = \sum_{x \in \mathrm{Range}(X)} \varphi(\{x,y\}) = \sum_{x \in \mathrm{Range}(X)} \mathbb{P}(X=x, Y=y)$$

Conditional distribution $\mathbb{P}(A) > 0$, $A \subseteq S$

$y \in \mathrm{Range}(Y)$ $\mathbb{P}(Y=y) > 0$

$Z := X \mid Y=y$ is the random variable describes X given that $\{Y=y\}$ has happened. The distribution of Z, is given by

$$\mathbb{P}(Z=z) = \mathbb{P}(X=z \mid Y=y) = \frac{\mathbb{P}(X=z, Y=y)}{\mathbb{P}(Y=y)}$$

$z \in \mathrm{Range}(X)$

$P(A) > 0$

≤ 5

> 0

n variable

$\{Y = y\}$

$= P(\{X = z\} \mid Y = y)$

$X = z, Y = y$

$P(Y = y)$

$z \in \mathrm{Range}(X)$

Memoryless Property

$m, n \in \mathbb{N}$ $\qquad X \sim \text{Geometric}(p) \qquad 0 < p < 1$

$A = \{X > m\}$

$$\mathbb{P}(A) = P(X > m) = \mathbb{P}\left(\bigcup_{k=m+1}^{\infty} \{X = k\}\right)$$

$$\doteq \sum_{k=m+1}^{\infty} \mathbb{P}(X = k) = \sum_{k=m+1}^{\infty} (1-p)^{k-1} p$$

$\alpha = p$

$$= p \sum_{k=m+1}^{\infty} (1-p)^{k-1} = p\,(1-p)^m \sum_{k=1}^{\infty} (1-p)^{k-1} = p(1-p)^m \frac{1}{1-(1-p)}$$

$r = (1-p)$

$$= (1-p)^m$$

Analysis I facts

$$\sum_{n=1}^{\infty} (\alpha a_n) = \alpha \sum_{n=1}^{\infty} a_n$$

$$\sum_{k=m+1}^{\infty} r^{k-1} = \frac{r^m}{1-r} \qquad 0 < r < 1$$

$$\sum_{k=1}^{\infty} b_k \doteq T_n = \sum_{k=1}^{n} b_k \quad \lim_{n\to\infty} T_n$$

$B = \{X > m+n\}$, $P(B) = (1-p)^{m+n}$ (Replace m by m+n in ⊛)

$$P(X > m+n \mid X > m) = P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

on $A \cap B = \{X > m, X > m+n\} = \{X > m+n\} = B$

$$= \frac{P(B)}{P(A)} = \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n = P(X > n)$$

i.e. Give

Probabil

the a

Then

— It the

variable

future

)

B

$= P(X > n)$

i.e. Given that no success occurred in the first m trials probability that a success will occur in next n trials is the probability that a success occurs in first n trials

This is called memoryless property of Geometric(p)

— If the waiting time in a queue (for an event) is modelled as a Geometric(p) random variable then no matter how long one has waited for a given event, the future waiting time is the same given that the event hasn't happened yet

Multinom

Consider

n-independ

Probabili

$X_j = \#$

$(X_1, \ldots, X_k)$

Multinomial distribution $(n, k; p_1, \ldots, p_k)$

$\sum_{j=1}^{k} p_j = 1$ | $k \geq 2$.

Consider an experiment where there are k possibilities, & we perform n-independent trials of the experiment. Let p_k be the Probability of outcome k

$X_j \equiv$ # of trials (among n-independent trial) in which j^{th} outcome occurs

$(X_1, \ldots, X_k)$

Observation $X_1 + X_2 + \ldots + X_k = n \Rightarrow \begin{cases} \text{- dependent} \\ \text{- Q: Find joint distribution of } (X_1, \ldots, X_k) \end{cases}$

$k=2$ (seen earlier)

$X_1 \in \{0,1,\dots,n\}$ $X_2 \in \{0,1,\dots,n\}$

$X_1 + X_2 = n$

$p_1 + p_2 = 1$

$$\mathbb{P}(X_1 = k,\ X_2 = j) = 0 \quad \text{if } j+k \neq n$$

$$\mathbb{P}(X_1 = k,\ X_2 = n-k) = \mathbb{P}\left(\text{“}k \text{ independent trials out of } n \text{ have outcome } 1\text{”}\right)$$

$$= {}^{n}C_{k}\,(p_1)^{k}\,(1-p_1)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!}\,p_1^{k}\,p_2^{n-k} \qquad k \in \{0,1,\dots,n\}$$

$$\boxed{\mathbb{P}(X_1 = x_1,\ X_2 = x_2) = \begin{cases} \dfrac{n!}{x_1!\,x_2!}\,p_1^{x_1}\,p_2^{x_2} & x_i \in \{0,1,\dots,n\},\ x_1 + x_2 = n \\ 0 & \text{otherwise} \end{cases}}$$

General case

x_i

$\mathbb{P}$

Assum

$B = \{w$

General case

$x_i \in \{0,1,\dots,n\} \quad 1 \le i \le k$

$$\mathbb{P}(X_1=x_1,\dots,X_k=x_k) = 0 \quad \text{if} \quad \sum_{i=1}^{k} x_i \neq n$$

Assume $\sum_{i=1}^{k} x_i = n$.

$$\mathbb{P}(X_1=x_1, X_2=x_2, \dots, X_k=x_k) = \mathbb{P}(B) = \sum_{w\in B} \mathbb{P}(\{w\})$$

$$B = \{w \in S \mid X_1(w)=x_1, X_2(w)=x_2, \dots, X_k(w)=x_k\}$$

$$w \in B \Rightarrow \mathbb{P}(\{w\}) = \prod_{i=1}^{k} p_i^{x_i}$$

$$\Rightarrow \mathbb{P}(B) = |B| \prod_{i=1}^{k} p_i^{x_i}$$

$k=4$

$$\mathbb{P}(\underbrace{\_\ \_\ \cdots\ \_}_{n}) \quad x_1, x_2, x_3, x_4$$

$$= p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

General case: $x_i \in \{0, 1, \ldots, n\}$ $\quad 1 \le i \le k$

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = 0 \quad \text{if} \quad \sum_{i=1}^{k} x_i \neq n$$

Assume $\sum_{i=1}^{k} x_i = n.$

$$P(X_1 = x_1, X_2 = x_2, \ldots X_k = x_k) = \mathbb{P}(B) = \sum_{w \in B} \mathbb{P}(\{w\})$$

$$B = \{ w \in S \mid X_1(w) = x_1, X_2(w) = x_2, \ldots, X_k(w) = x_k \}$$

$$w \in B \Rightarrow \mathbb{P}(\{w\}) = \prod_{i=1}^{k} p_i^{x_i}$$

$$\Longrightarrow \mathbb{P}(B) = |B| \prod_{i=1}^{k} p_i^{x_i}$$

$k = 4$

$$\mathbb{P}(\overbrace{\_\,\_\,\_}^{x_1}\,\underbrace{\_\,\_\,\_}_{x_2}\,\overbrace{\_\,\_\,\_}^{x_3}\,\underbrace{\_\,\_}_{x_4}) \quad n$$

$$= p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

- $B \equiv$ allocating n balls into k boxes such that box j has x_j balls.

$$= \frac{n!}{\prod_{i=1}^{k} x_i!} \quad \text{if } \sum_{i=1}^{k} x_i = n$$

So,

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = \begin{cases} \frac{n!}{\prod_{i=1}^{k} x_i!} \prod_{i=1}^{k} p_i^{x_i} & \text{where } x_i \in \{0, 1, \ldots, n\} \ \& \ \sum_{i=1}^{k} x_i = n \\ 0 & \text{otherwise} \end{cases}$$

Multinomial $(n, k, p_1 \ldots p_k)$

$X_j \equiv$ # of trials (among n-independent trials) in which j^{th} outcome occurs

$(X_1, \ldots, X_k)$ **observation** $X_1 + X_2 + \ldots + X_k = n \Rightarrow \begin{cases} \text{- dependent} \\ \text{- Q: Find joint distribution of } (X_j, X_k) \end{cases}$

3.3 Functions of Random Variables

- Random variables are already functions

$$X : S \longrightarrow \mathbb{R}$$

It turns out that in many cases it is worthwhile to understand functions of random variables

X – "length" of a rectangle
Y – breadth of a "

$X \sim \text{Uniform}\{1,2,3,4,5\}$
$Y \sim \text{Uniform}\{1,2,3,4,5\}$

$A = X \cdot Y$ – area of the "random" rectangle constructed

Exa

General Question: $X_1, \ldots, X_n$ are a collection of random variables

$f : \mathbb{R}^n \to \mathbb{R}$ $\qquad Z = f(x_1, \ldots, x_n)$ Distribution of Z?

$f : \mathbb{R} \to \mathbb{R}$ & $X : S \to T$ a discrete random variable

$\Rightarrow$ $Y : S \to U$ $\qquad Y = f(X)$ is also a discrete random variable

$\mathbb{P}(Y = y) = \mathbb{P}(f(X) = y)$ for all $y \in U$

Key to understanding distribution of Y

Example 3.3.11:

$X \sim \text{Uniform}(\{-2,-1,0,1,2\})$

$f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

$Y = f(X) \quad \text{Range}(Y) = \{0,1,4\}$

$\mathbb{P}(Y=0) = \mathbb{P}(X^2=0) = \mathbb{P}(X=0) = 1/5$

$\mathbb{P}(Y=1) = \mathbb{P}(X^2=1) = \mathbb{P}(X=-1 \cup X=1) = \mathbb{P}(X=-1) + \mathbb{P}(X=1) = 2/5$

$\mathbb{P}(Y=4) = \mathbb{P}(X^2=4) = \mathbb{P}(X=-2 \cup X=2) = \mathbb{P}(X=-2) + \mathbb{P}(X=2) = 2/5$

Example 3.3.2 We roll a dice three times

- Let X_i be the outcome of i^{th} role

$$Y = X_1 + X_2 + X_3$$

$\mathrm{Range}(X_1) \times \mathrm{Range}(X_2) \times \mathrm{Range}(X_3)$

$f : \mathbb{R} \to \mathbb{R}$

$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$

$Y = f(X_1, X_2, X_3)$

$\mathrm{Range}(Y) = \{3, 4, 5, 6, \ldots, 18\}$

$P(Y = 6) = ?$

$\{Y = 6\} \equiv$ can happen in 3 ways

Case I : $A_1 = \{X_1 = 2, X_2 = 2, X_3 = 2\}$

independence

$$\mathbb{P}(X_1 = 2, X_2 = 2, X_3 = 2) = P(X_1 = 2)\, P(X_2 = 2)\, P(X_3 = 2)$$

$$= 1/6^3$$

$X_i \sim$ uniform $\{1, \ldots, 6\}$

Case II : $A_2 = \{X_1 = 3, X_2 = 1, X_3 = 2\} \cup \{X_1 = 2, X_2 = 3, X_3 = 1\} \cup \{X_1 = 3, X_2 = 2, X_3 = 1\}$

$\cup \{X_1 = 2, X_2 = 1, X_3 = 3\} \cup \{X_1 = 1, X_2 = 2, X_3 = 3\} \cup \{X_1 = 1, X_2 = 3, X_3 = 2\}$

$$\mathbb{P}(A_2) = 6 \cdot \frac{1}{6^3}$$

Events mutually exclusive

& X_1, X_2, X_3 are independent & $X_i \sim$ uniform $\{1, \ldots 6\}$

Case 3

$\mathbb{P}(Y = $

Case I : $A_1 = \{X_1 = 2,\ X_2 = 2,\ X_3 = 2\}$

independence

$$\mathbb{P}(X_1=2, X_2=2, X_3=2) = P(X_1=2)\,P(X_2=2)\,P(X_3=2)$$

$$= 1/6^3$$

$X_i \sim \text{uniform}\{1, \dots, 6\}$

Case II : $A_2 = \{X_1=3, X_2=1, X_3=2\} \cup \{X_1=2, X_2=3, X_3=1\} \cup \{X_1=3, X_2=2, X_1=1\}$

$\cup \{X_1=2, X_2=1, X_3=3\} \cup \{X_1=1, X_2=2, X_3=3\} \cup \{X_1=1, X_2=3, X_3=2\}$

$$\mathbb{P}(A_2) = 6 \cdot \frac{1}{6^3}$$

events mutually exclusive

& X_1, X_2, X_3 are independent & $X_i \sim \text{Uniform}\{1, \dots 6\}$

Case 3

$\mathbb{P}(Y = $

<u>Case 3</u> :

$A_3 = \{X_1 = 4, X_2 = 1, X_3 = 1\} \cup \{X_1 = 1, X_2 = 4, X_3 = 1\} \cup \{X_1 = 1, X_2 = 1, X_3 = 4\}$

$\mathbb{P}(A_3) = 3 \cdot \frac{1}{6^3}$

$\mathbb{P}(Y = 6) = \mathbb{P}(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

$= \frac{1 + 6 + 3}{6^3} = 10/216$

$X_1 = 1\}$

$3, X_3 = 2\}$

Example 3.3.3 : let $X \sim \text{Bernoulli}(p)$

- let Y be independent of X & $Y \sim \text{Bernoulli}(p)$

$Z = X+Y$

- $P(Z=0) = P(X+Y=0) \overset{\text{description of event}}{=} P(X=0, Y=0) \underset{\text{independence}}{=} P(X=0)P(Y=0) = (1-p)^2$
- $P(Z=1) = P(X+Y=1) = P(X=1, Y=0 \lor X=0, Y=1) =$

(disjoint events) $= P(X=$

- $P(Z=$

Ex. X_1 ,

Exampl

(disjoint events) $= P(X=1, Y=0) + P(X=0, Y=1) = P(X=1)P(Y=0) + P(X=0)P(Y=1) = 2p(1-p)$.

$P(Z=2) = P(X=1, Y=1) = P(X=1)P(Y=1) = p^2$

$\therefore Z \sim \text{Binomial}(2, p)$

Ex.- $X_1, \ldots X_n$ are i.i.d Bernoulli$(p) \Rightarrow Z = X_1 + \ldots + X_n \sim$ Binomial(n, p)

(induction & $Y \sim$ Binomial(n, p) & $W \sim$ Bernoulli(p) independent $\Rightarrow Y + W \sim$ Binomial$(n+1, p)$)

Example 3.3.4

$X \sim \text{Poisson}(\lambda_1)$ $Y \sim \text{Poisson}(\lambda_2)$ $Z = X + Y$

& independent

$P(X=k)$
$P(Y=k)$
$e^{-\lambda}\lambda^k / k!$

$\mathrm{Ran}_{X}(Z) = \{0\} \cup \mathbb{N}$

$$\mathbb{P}(Z=0) = P(X+Y=0) = P(X=0,\ Y=0) = P(X=0)\,P(Y=0) = e^{-\lambda_1} e^{-\lambda_2}$$
$$= e^{-(\lambda_1+\lambda_2)}$$

$n \in \mathbb{N}$ $\quad \mathbb{P}(Z=n) = \mathbb{P}(X+Y=n) = \mathbb{P}\left(\bigcup_{j=0}^{n} \{X=j,\ Y=n-j\}\right) = \sum_{j=0}^{n} P(X=j)\,P(Y=n-j)$

$$= \sum_{j=0}^{n} \frac{e^{-\lambda_1}\lambda_1^{j}}{j!}\,\frac{e^{-\lambda_2}\lambda_2^{n-j}}{(n-j)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\,\lambda_1^{j}\,\lambda_2^{n-j}$$

$$(\lambda_1+\lambda_2)^n = \sum_{j=0}^{n} \binom{n}{j} \lambda_1^{j} \lambda_2^{n-j}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

$$\boxed{Z \sim \mathrm{Poisson}(\lambda_1+\lambda_2)}$$

$X \sim \text{Poisson}(\lambda_1)$ independent

$Y \sim \text{Poisson}(\lambda_2)$

$Z = X + Y$ $\quad Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$

$W := X \,\big|\, Z = n$ conditional distribution of X given $Z = n$

"Range(W)" $= \{0, 1, \dots n\}$

$$\boxed{\mathbb{P}(X = k \mid Z = n) = 0 \ \forall k \geq n+1}$$

$j \in \{0, 1, \ldots, n\}$

$$P(X=j \mid Z=n) = \frac{P(X=j, Z=n)}{P(Z=n)} = \frac{P(X=j, Y=n-j)}{P(Z=n)} = \frac{P(X=j)\,P(Y=n-j)}{P(Z=n)}$$

$$= \frac{\dfrac{\cancel{e^{-\lambda_1}}\lambda_1^{j}}{j!} \cdot \dfrac{\cancel{e^{-\lambda_2}}\lambda_2^{n-j}}{(n-j)!}}{\dfrac{\cancel{e^{-(\lambda_1+\lambda_2)}}(\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{j!\,(n-j)!}\,\frac{\lambda_1^{j}\,\lambda_2^{n-j}}{(\lambda_1+\lambda_2)^n}$$

$$= {}^{n}C_{j}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{j}\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-j} = {}^{n}C_{j}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{j}\left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-j}$$

$$\boxed{X \mid Z=n \sim \text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)}$$

$n \in \mathbb{N}$

$Z \sim$

$n+1$