

Continuous random Variables

- we have discarded discrete random variables.

$$X: S \rightarrow T \quad T \subseteq \mathbb{R}, \quad T \text{ - countable.}$$

$\{P(X=t)\}_{t \in T}$ - completely characterized X .
 $P(X=t) > 0 \Rightarrow t \in T$

$$P(X \in A) = \sum_{t \in A} P(X=t)$$

$$\{1, 2, 3, \dots, 10\}$$

$$\underline{I = [0, 1]}$$

Suppose we want to choose a number "uniformly" from I .

• In discrete setting, choose a number uniformly from $\{1, \dots, N\}$

$$P(\{i\}) = \frac{1}{N} \quad \forall i \in \{1, \dots, N\}$$

• Want a probability such that

$$x \in I \quad P(\{x\}) = p \quad 0 \leq p \leq 1$$

Suppose it exists $E = \{1/n \mid n \geq 1\} \subseteq I$

$$P(E) = \sum_{n=1}^{\infty} P(\{1/n\}) = \sum_{n=1}^{\infty} p$$

Axiom

make sense

$$(p=0) \Rightarrow P(E) < \infty \quad \text{---} \textcircled{+}$$

$$1 = P(I) = P\left(\bigcup_{x \in I} \{x\}\right) \stackrel{\text{Axiom 2}}{\underset{\substack{\text{not governed} \\ \text{by Axiom 2}}}{=}} \sum_{x \in I} P(\{x\}) = 0 \quad - \textcircled{A}$$

I - uncountable $\equiv$ set of outcomes, this implies that Axiom 2 will not apply in $\textcircled{A}$ to get a contradiction but $\textcircled{A}$ $\checkmark$ to yield $p=0$.

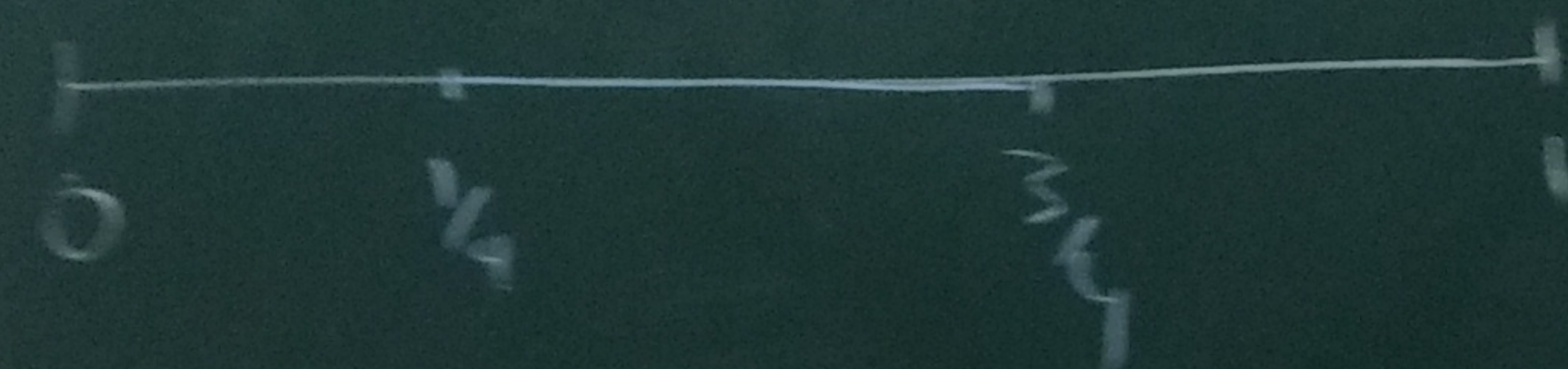
$\textcircled{Q}$: Is it possible to have $P(\{x\}) = 0 \quad \forall x \in I$ & yet $P(I) = 1$?

$\textcircled{Q}$: How does one choose a number "uniformly" from I ?

$\boxed{\tilde{\varphi}}$

An approach

$$P(I) = 1$$



$$P\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) = \frac{\text{length}\left[\frac{1}{4}, \frac{3}{4}\right]}{\text{length of } I} = \frac{\frac{3}{4} - \frac{1}{4}}{1} = \frac{1}{2}$$

Def: $A \subseteq [0, 1]$ assign $P(A) = \text{"length" of } A$

Next difficulty: "length" of A is easy to define when A is an interval
 A is disjoint union of intervals

KNOWN : one cannot define length for $A \subseteq [0,1]$
all

- Need a different definition of an event.

keep the following in mind, while defining events :-

I - entire sample space is an event

- all countable union of events to be events

- complement of events to be an event

intervals should be events
for choosing "uniformly" from I

Examples -

Definition 5.1.1 (σ -field) let S be a sample space

$\mathcal{P}(S) \supseteq \mathcal{F}$ - a collection of subsets of S is a σ -field if

$$\textcircled{1} S \in \mathcal{F}$$

$$\textcircled{2} A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$\textcircled{3} \{E_n\}_{n \geq 1}, E_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$$

(unlike axiom of probability, NO requirement of disjoint)

Examples:- $\mathcal{F} = \{S, \emptyset\}$, $\mathcal{F} = \{S, A, A^c, \emptyset\}$...

$S = \mathbb{R}$, there is a smallest σ -field that contains all intervals, it is called the

BOREL σ -field. — $\mathcal{B}$.

It so happens for $A \in \mathcal{B}$, one can define a notion of length.

— We need a "slightly" new definition of probability

Definition 5.1.2 [Probability Space Axioms]

S - sample space $\mathcal{F}$ - σ -field on S

A probability $\mathbb{P}$ on S is a function $\mathbb{P}: \mathcal{F} \rightarrow [0, 1]$

such that

$$(1) \mathbb{P}(S) = 1$$

$$(2) \{E_n\}_{n=1}^{\infty} \quad E_n \in \mathcal{F} \text{ \& } E_n \cap E_m = \emptyset \quad m \neq n \text{ then}$$

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

$$(S, \mathcal{F}, \mathbb{P}) \in \text{Probability on } S$$

All our previous work is still rigorous that

S countable in discrete setting. The

Ex: $\mathcal{F} = \mathcal{P}(S)$ is also a σ -field
(i.e. $\emptyset, \mathcal{C}, \mathcal{C}$ & definition 5.1.1. hold)

Remarks:-

S uncountable, in many cases one
cannot work with $\mathcal{F} = \mathcal{P}(S)$

$S = \mathbb{R}$, one works with $\mathcal{F} \subseteq \text{Borel } \sigma\text{-field}$.
Understanding Borel σ -field is beyond the scope
of this class. An instance $\mathcal{F} \subseteq \mathcal{P}(S)$ & $A \notin \mathcal{F}$

What we

key

What we will do is: $S = \mathbb{R}$, $\mathcal{F} = \text{event space} = \text{Borel } \sigma\text{-field}$

- Think of events in S as $\mathbb{R}$ = interval, ^{Countable & finite} union of intervals, & complements $x \in \mathbb{R}$

- all our results from 12th class calculus.

$$\{x\} = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n})$$

$$= \left(\bigcup_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n}) \right)^c$$

$$= \left(\bigcup_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n}) \bigcup_{n=1}^{\infty} [x + \frac{1}{n}, \infty) \right)^c$$

Key difficulty: One cannot define $P: \mathcal{F} \rightarrow [0,1]$

by specifying $P(\{x\}) \forall x \in S$

all $x \in \mathbb{R}$
 $\{x\} \in \mathcal{F}$

because by countable argument $P(\{x\}) = 0$

Probability density function

(Let S be a sample space)

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$S = \mathbb{R}, \quad \mathcal{F} = \text{Borel } \sigma\text{-field}$$

$$A \subseteq \mathcal{F} \rightarrow P(A) = \int_A f(x) dx \Rightarrow$$

$$A = [a,b] \subseteq [0,1]$$

$$P([a,b]) = b-a$$

$$P([0,1]) = 1$$

$$P(\{1/2\}) = 0$$

$\Rightarrow$ Probability
density
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