

Recall

$X_1, \dots, X_n$ are i.i.d. Bernoulli (p)

$$Y_n = \sum_{j=1}^n X_j, \text{ we know } Y_n \sim \text{Binomial}(n, p)$$

for
 $1 \leq j \leq n$

$$P(X_j = 1 \mid Y_n = k) = \frac{P(X_j = 1, Y_n = k)}{P(Y_n = k)}$$

$$P(X_j = 1) = p$$

$$= \frac{p \binom{n-1}{k-1} p^{k-1} (1-p)^{n-1-(k-1)}}{\binom{n}{k} p^k (1-p)^{n-k}} = k/n$$

Definition

Let

A

Consider

(Event)

Conditional distributions

Definition 3.2.5 X is a discrete random variable on a sample space S equipped with probability P .

$A \subseteq S$, A -event such that $P(A) > 0$

Consider $\Phi: \mathcal{F} \rightarrow [0,1]$

$$\Phi(B) = P(X \in B | A)$$

(Event space)

Φ - conditional distribution of X given A

As discussed $\phi(\cdot)$ is determined entirely by

$$(*) \text{ ————— } \phi(a) = P(X=a|A) \quad \forall a \in \text{Range}(X)$$

$$\left(\phi(B) = \sum_{a \in B} \phi(a) \right)$$

(*) — enough to describe the conditional distribution of X given A .

Usually, the events " A " that we be interested in are those associated with another dependent random variable.

Say distribution of " X given $Y=b$ " $\equiv$ (denoted by) $\equiv X|Y=b$

Example 3.26

$X \sim \text{uniform } \{1, 2\}$

$\text{Range}(X) = \{1, 2\}$

$Y = \# \text{ of heads in } X \text{ tosses of a coin with probability } 1/2$

$\text{Range}(Y) = \{0, 1, 2\}$

$$Y=2 \Rightarrow X=2$$

$$Y=0 \Rightarrow X=1 \text{ or } X=2$$

$$A = \{X=1\}$$

$$P(A) = 1/2 > 0$$

Conditional distribution of Y given A

$$P(Y=a | A) \quad \forall a \in \text{range}(Y)$$

$$\mathbb{L} \begin{cases} P(Y=0 | A) \\ P(Y=1 | A) \\ P(Y=2 | A) \end{cases}$$

$$a=0$$

$$a=1$$

$$a=2$$

But

$$P(Y=2 | X=1) = 0$$

$$P(Y=0 | X=1) = 1-p$$

$$P(Y=1 | X=1) = p$$

$Y | X=1$ only takes values $\{0, 1\}$ and $Y | X=1 \sim \text{Bernoulli}(p)$

$$B = \{X=2\} \quad P(B) = \frac{1}{2} > 0$$

$$Y|B = Y|X=2$$

$$Y|X=2 \text{ takes values } \{0, 1, 2\}$$

$$P(Y=0|X=2) = {}_2C_0 p^0 (1-p)^2 = (1-p)^2$$

$$P(Y=1|X=2) = {}_2C_1 p^1 (1-p)^1 = 2p(1-p)$$

$$P(Y=2|X=2) = {}_2C_2 p^2 (1-p)^0 = p^2$$

$$\Rightarrow Y|X=2 \sim \text{Binomial}(2, p)$$

In the previous example the dependency of Y on X was clearly stated in terms of conditional distributions. This may always not be the case

A more (complete) general description of the dependency is probabilities of all combinations of values taken by X & Y

$$P(X=x, Y=y) \quad \text{where } x \in \text{Range}(X) \quad y \in \text{Range}(Y)$$

Definition

var

Prob

This means

Joint dist

Definition 3.27 (Joint distribution) ... let X & Y be discrete random

variables. The "joint distribution" of X & Y is the

Probability $\mathbb{P}$ on pairs of values from the range(X) & range(Y)

defined by

$$\mathbb{P}(\{a, b\}) = \mathbb{P}(X=a, Y=b).$$

This may be extended to a finite collection of random variables $X_1, \dots, X_n$

Joint distribution of $X_1, \dots, X_n$ is given by $\mathbb{P}(\{a_1, \dots, a_n\}) = \mathbb{P}(X_1=a_1, \dots, X_n=a_n)$ $a_i \in \text{range}(X_i)$ $1 \leq i \leq n$

Ex

$X \sim$

$Y =$

Joint

$$\mathbb{P}\left(\bigcap_{j=0}^3 X=j, Y=j\right) =$$

$$+ \mathbb{P}(X=1, Y=2)$$

$$+ \mathbb{P}(X=1, Y=1)$$

$$+ \mathbb{P}(X=1, Y=0)$$

Example 3.2.8 (Example 3.2.8 - revisited)

$$X \sim \text{Uniform}\{1, 2\}$$

$Y = \# \text{ of heads in } X \text{ tosses of a fair coin } (p = 1/2)$

Joint distribution of two random variables is given by a table

$$P(\bigcup_{j=0}^3 X=1, Y=j) = P(X=1)$$

$$+ P(X=1, Y=2)$$

$$+ P(X=1, Y=1)$$

$$+ P(X=1, Y=0)$$

	$X=1$	$X=2$	
$Y=0$	$1/4$	$1/8$	$3/8$
$Y=1$	$1/4$	$1/4$	$1/2$
$Y=2$	0	$1/8$	$1/8$
	$1/2$	$1/2$	

$$P(X=1, Y=0) = P(Y=0|X=1) P(X=1) = 1/4$$

$$P(X=2, Y=0) = P(Y=0|X=2) P(X=2) = 1/8$$

$$P(X=1, Y=1) = P(Y=1|X=1) P(X=1) = 1/4$$

$$P(X=2, Y=1) = P(Y=1|X=2) P(X=2) = 1/4$$

$$P(X=2, Y=2) = P(Y=2|X=2) P(X=2) = 1/8$$

$1 \leq i \leq n$
 $c(X_i)$

In the previous table, the entire row of Y or X of the for

$$P(X=i, Y=j) \quad i \in \{0,1\}$$

$$j \in \{0,1,2\}$$

Find Row-sum

$$3/8 = P(X=1, Y=0) + P(X=2, Y=0)$$

Disjoint events
Thm 1.14 (ii)

$$= P(X=1, Y=0 \cup X=2, Y=0)$$

$$= P(\{X=1 \cup X=2\}, Y=0)$$

$$= P(S \cap \{Y=0\}) = P(Y=0)$$

Row sum
give corresponding
"Y-probabilities"

$$2^{nd} - 0$$

$$1/2$$

$$X \sim \mathcal{E}$$

$$X \sim \mathcal{U}$$

2nd - Column Sum

$$1/2 = P(X=2, Y=0) + P(X=2, Y=1) + P(X=2, Y=2)$$

$$= P(X=2, Y=0 \cup X=2, Y=1 \cup X=2, Y=2)$$

$$= P(X=2, \bigcup_{j=0}^2 Y=j)$$

$$= P(X=2 \cap S) = P(X=2)$$

(column - sum
give corresponding
"X-Probabilities")

X & Y are legitimate discrete random variables

$X \sim \text{uniform}\{1, 2\}$. — (Joint distribution) — $P(Y=0) = 3/8$ $P(Y=1) = 1/8$ $P(Y=2) = 1/8$ (distribution of Y)

Further

$P(X=1,$

Joint

$P(X=1, Y=0) = P$

$P(X=1, Y=1)$

$P(X=1, Y=2)$

$P(X=1, Y=0)$

(Y=0)

Further

$$P(X=1|Y=1)$$

Bayes
Theorem

$$P(X=1, Y=1)$$

$$P(Y=1)$$

$$= \frac{1/4}{1/2} = 1/2$$

(table)

definition

$$P(X=1, Y=0) = 1/4 \neq 3/16 = P(X=1) P(Y=0) \Rightarrow X \text{ \& Y are NOT independent}$$

Joint distribution of two random variables is given by a table

$$P(\bigcup_{j=0}^3 X=1, Y=j) = P(X=1)$$

$$+ P(X=1, Y=2)$$

$$+ P(X=1, Y=1)$$

$$+ P(X=1, Y=0)$$

	X=1	X=2	
Y=0	1/4	1/8	3/8
Y=1	1/4	1/4	1/2
Y=2	0	1/8	1/8
	1/2	1/2	

$$P(X=1, Y=0) = P(Y=0|X=1) P(X=1) = 1/4$$

$$P(X=2, Y=0) = P(Y=0|X=2) P(X=2) = 1/8$$

$$P(X=1, Y=1) = P(Y=1|X=1) P(X=1) = 1/4$$

$$P(X=2, Y=1) = P(Y=1|X=2) P(X=2) = 1/4$$

$$P(X=2, Y=2) = P(Y=2|X=2) P(X=2) = 1/8$$

Suppose X, Y are two discrete random variables

Joint with joint distribution $\phi(x, y) = P(X=x, Y=y)$

then distribution of X is given by

$P(X=x) = \sum_{y \in \text{Range}(Y)} P(X=x, Y=y) = \sum_{y \in \text{Range}(Y)} \phi(x, y)$

Marginal distribution of X & Y

the distribution of Y is given by

$$P(Y=y) = \sum_{x \in \text{Range}(X)} P(X=x, Y=y) = \sum_{x \in \text{Range}(X)} \phi(x, y)$$

$X \sim Y$

$X \sim \text{unif}$