

Determines how likely an event
is going to occur?

= Prediction becomes a key
objective.

First objective:

What is possible?

- Understand all possible outcomes.

Three items

Sample Space

Experiment

Outcome,
Event

event

- Final: - Probability

Definition 1.1.1 (Sample Space) A sample space S is a set

All elements of S are outcomes [think of S as a list of all possible outcomes]

An experiment is the process of selecting one outcome from S .

Examples: $S = \{1, 2, 3, 4, 5, 6\}$

Experiment: - Roll a die & note down the outcome on the top face.

outcomes
event

$$S = \{H, T\}$$

Experiment: Toss a coin & record the outcome

$$S = \{ \text{Int of all countries} \}$$

Experiment: Kabaddi tournament winner

Note:- In the first two situations all outcomes were equally likely. In the third, some countries ^{may} play Kabaddi better than others.

ation of an event)

subset of S

in to

age of

{5,6}

Note: - $S, \emptyset$ are legitimate events

- we will later see that all possible subsets need not be events
(Distraction)

Next Goal - To understand how to assign value to every event, called Probability

Definition 1.1.3 [Pr]

S - sample space

All "Probability"

TP:

①

P

②

(E, E)

1.1.2. (Temporary) Definition of an Event

E is any subset of S
($E \subseteq S$)

definition allow us to
how likely a range of
are.

$\{1, 2, \dots, 6\}$

larger than 4 $\} = \{5, 6\}$

Note: - S, ϕ are legitimate events
- we will later see that all
possible subsets need not be events
(Distraction)

Next Goal - To understand
how to assign value to every
event, called Probability

Definition 1.1.3 [Probability Space Axioms]

S - sample space $\mathcal{F}$ - collection

A "Probability" is a function

$$P: \mathcal{F} \rightarrow [0, 1] = \{r \mid 0 \leq r \leq 1\}$$

$$\textcircled{1} P(S) = 1$$

$$\textcircled{2} (E_1, E_2, E_3, \dots, E_n) \text{ be a collection of disjoint events}$$

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} P(E_j)$$

Definition 1.12 (Temporary Definition of an Event)
 An event E is any subset of S
 $(E \subseteq S)$

The above definition allows us to understand how likely a range of possibilities are.

Example $S = \{1, 2, \dots, 6\}$

$E = \{\text{outcome is larger than 4}\} = \{5, 6\}$

Note: -

- we will
possible S

Next Goal

how to
event,

Definition 1.13 (Axioms of Probability)

S - sample space

$\mathcal{F}$ - collection of events

$P: \mathcal{F} \rightarrow [0, 1]$

is called "Probability"

if ① $P(S) = 1$

② $\{E_n\}_{n \geq 1}, E_n \in \mathcal{F}$ for all $n \geq 1$

& $E_l \cap E_m = \emptyset$ for $m \neq l$

Kolmogorov

Definition 1.12

then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

$$T_n = \sum_{m=1}^n \mathbb{P}(E_m)$$

$$\lim_{n \rightarrow \infty} T_n = T = \sum_{n=1}^{\infty} \mathbb{P}(E_n)$$

always exist

$$\left\{x \in S \mid x \in E_n \text{ for some } n \geq 1\right\}$$

Basic Properties

Theorem 1.4

Let $\mathbb{P}$ be a probability on a sample space S (i.e. in definition 1.3)
then

$$(i) \mathbb{P}(\phi) = 0$$

(ii) $n \geq 1$, $\{E_k\}_{k=1}^n$ is a collection of disjoint events. Then

$$\mathbb{P}\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mathbb{P}(E_k)$$

(iii) E, F are two events $E \subseteq F \Rightarrow \mathbb{P}(E) \leq \mathbb{P}(F)$

E, F are 2 events, $E \subseteq F$

$$(iv) \mathbb{P}(E^c \cap F) = \mathbb{P}(F) - \mathbb{P}(E)$$

$$(v) \mathbb{P}(E^c) = 1 - \mathbb{P}(E) \quad \text{or} \quad \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

(vi) $E \& F$ are two events.

Answer 1.1.2

$$E_k = \emptyset \quad \forall k \geq 1$$

$$E_k = \emptyset$$

Answer 2 Definition 1.1.3

$$P(\emptyset) = \sum_{k=1}^{\infty} P(E_k) \Leftrightarrow P(\emptyset) = 0$$

$$T_n = \sum_{k=1}^n P(\emptyset) = n P(\emptyset)$$

$$\lim_{n \rightarrow \infty} T_n = T \Leftrightarrow P(\emptyset) = 0 \quad \left. \vphantom{\lim_{n \rightarrow \infty} T_n = T} \right\} \text{E. Real Analysis}$$

$$(ii) \{E_k\}_{k=1}^n$$

$$E_m \cap E_l = \emptyset$$

$$1 \leq m \neq l \leq n$$

$$E_p = \emptyset \quad \forall p \geq n+1$$

$\Rightarrow \{E_m\}_{m=1}^{\infty}$ is a sequence of disjoint events

$$\bigcup_{m=1}^{\infty} E_m = \bigcup_{m=1}^n E_m \quad (*)$$

$$T_k = \sum_{i=1}^k P(E_i) \Rightarrow T_k = \sum_{i=1}^n P(E_i) \quad \forall k \geq n \quad (\text{by (ii) } P(E_i) = 0 \text{ for } i \geq n+1)$$

$$\Rightarrow T = \lim_{k \rightarrow \infty} T_k = \sum_{i=1}^n P(E_i) \quad (**)$$

Proof.

$$(i) E_k = \phi \quad \forall k \geq 1$$

$$\bigcup_{k=1}^{\infty} E_k = \phi$$

$\Rightarrow$ Axiom (2) Definition 1.1.3

$$P(\phi) = \sum_{k=1}^{\infty} P(E_k) \Leftrightarrow P(\phi) = 0$$

$$T_n = \sum_{k=1}^n P(\phi) = n P(\phi)$$

$$\lim_{n \rightarrow \infty} T_n = T \quad (=) P(\phi) = 0$$

} Ex. Real Analysis

$$(ii) \{E_k\}$$

$$\Rightarrow \{E_k\}$$

$$B_n \begin{matrix} \uparrow \\ \downarrow \end{matrix} \begin{matrix} \uparrow \\ \downarrow \end{matrix} \begin{matrix} \uparrow \\ \downarrow \end{matrix}$$

$$(ii) \quad \{E_k\}_{k=1}^n \quad E_m \cap E_l = \emptyset \quad 1 \leq m \neq l \leq n$$

$$E_p = \emptyset \quad \forall p \geq n+1$$

$\Rightarrow \{E_m\}_{m=1}^\infty$ is a sequence of disjoint events

$$\sum_{m=1}^{\infty} E_m = \bigcup_{m=1}^{\infty} E_m \quad (*)$$

$$\text{By } T_k = \sum_{i=1}^k P(E_i) \Rightarrow T_k = \sum_{i=1}^n P(E_i) \quad \forall k \geq n \quad (\text{by (ii) } P(E_p) = P(\emptyset) = 0 \quad \forall p \geq n+1)$$

$$\Rightarrow T = \lim_{k \rightarrow \infty} T_k = \sum_{i=1}^{\infty} P(E_i) \quad (**)$$

$$(*) \text{, } (**) \in \text{Axiom (2) \& Definition 1.1.3} \Rightarrow P\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n P(E_i)$$

(iii) E, F two events. $E \subseteq F$

$$E_1 = E, E_2 = F \cap E^c \Rightarrow E_1 \cap E_2 = \emptyset \text{ \& by (ii)}$$

$E \cap F$

$$P(E \cup E_2) = P(E_1) + P(E_2)$$

$$P(F) = P(E) + P(F \cap E^c)$$

xxx

$$\text{As } P(F \cap E^c) \geq 0 \Rightarrow P(F) \geq P(E)$$

(iv) Use $x \times x$ to get

$$P(F \cap E^c) \leq P(F) - P(E)$$

(v) Take $F = S$ in $x \times x$

$$P(S) = P(E) + P(E^c)$$

$$\Rightarrow 1 = P(E) + P(E^c) \Rightarrow \boxed{P(E^c) = 1 - P(E)}$$

Answer
Defn 1.1

(vi)

let

(vi) E, F are two events

$$E \cup F = \underbrace{E}_{E_1} \cup \underbrace{(F \cap E^c)}_{E_2}$$

$$\Rightarrow E_1 \cap E_2 = \emptyset \quad \text{by (ii)} \quad P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

$$P(E \cup F) = P(E) + P(F \cap E^c) - (+)$$

let $G_1 = F \cap E^c$ $G_2 = F$

$$G_1 \subseteq G_2 \quad \text{by (iv)}$$

$$P(G_2 \cap G_1^c) = P(G_2) - P(G_1) - (++)$$

$-P(E)$

$$G_2 \cap G_1^c = F \cap (F \cap E^c)^c = F \cap (E \cup F^c) = F \cap E$$

(Ex.)

++ $\Rightarrow$

$$P(F \cap E) = P(F) - P(F \cap E^c)$$

$$\Rightarrow P(F \cap E^c) = P(F) - P(E \cap F) \quad \text{---} \text{+++}$$

+

2 +++

$$P(E \cup F) = P(E) + P(F) - P(F \cap E)$$

Example 1.1.5

We toss a coin $\times \times \times$ to get Head or Tail

We call a coin "fair" if the outcome Head or Tail is equally likely.

(Definition 1.1.3 & Theorem 1.1.4 help in formulating $(S, \mathcal{F}, \mathbb{P})$)

$$S = \{H, T\}$$

$$\mathcal{F} = \mathcal{P}(S) = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$$

We

$$P: \mathcal{F} \rightarrow [0,1]$$

We want $P(\{H\}) = P(\{T\}) = p$

Axiom ① $\Rightarrow 1 = P(S)$

$$= P(\{H, T\})$$

$$= P(\{H\} \cup \{T\})$$

Theorem 1.1.4 (iii) $\Rightarrow P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}) = p + p = 2p$

$$S = \{H, T\}$$

$$\mathcal{F} = \{\emptyset, \{H\}, \{T\}, S\}$$

$$P: \mathcal{F} \rightarrow [0,1]$$

$$P(\emptyset) = 0, \quad P(\{H\}) = P(\{T\}) = \frac{1}{2}, \quad P(S) = 1$$

$$S = \{H, T\}$$

$$\mathcal{F} = \{\emptyset, \{H\}, \{T\}, S\}$$

$$P: \mathcal{F} \rightarrow [0, 1]$$

$$P(\emptyset) = 0$$

$$P(\{H\}) = P(\{T\}) = \frac{1}{2}$$

$$P(S) = 1$$

$$\Rightarrow p = \frac{1}{2}$$

$$\{T\})$$

$$H \cup \{T\}$$

$$P(H) + P(T) = p + p = 2p$$