

Recall X - be a discrete random variable

$g: \mathbb{R} \rightarrow \mathbb{R}$
 $Y = g(X)$ } Y is also a discrete random variable

$$T = \text{Range}(Y) \quad P(Y=t) = \sum_{\{u \in \text{Range}(X) : g(u)=t\}} P(X=u)$$

(distribution of Y)

X is a continuous random variable
 $g: \mathbb{R} \rightarrow \mathbb{R}$

$Y = g(X)$ (no clear procedure)

$$g(x) = \begin{cases} 2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$Y \sim \text{Bernoulli}(\frac{1}{2})$$

$$X \sim \text{Normal}(0,1)$$

$$X \sim \text{Uniform}(0,1)$$

$$Y = g(X) = X^2 \quad Y = g(X)$$

Y is a continuous random variable
with pdf

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 5.2.8

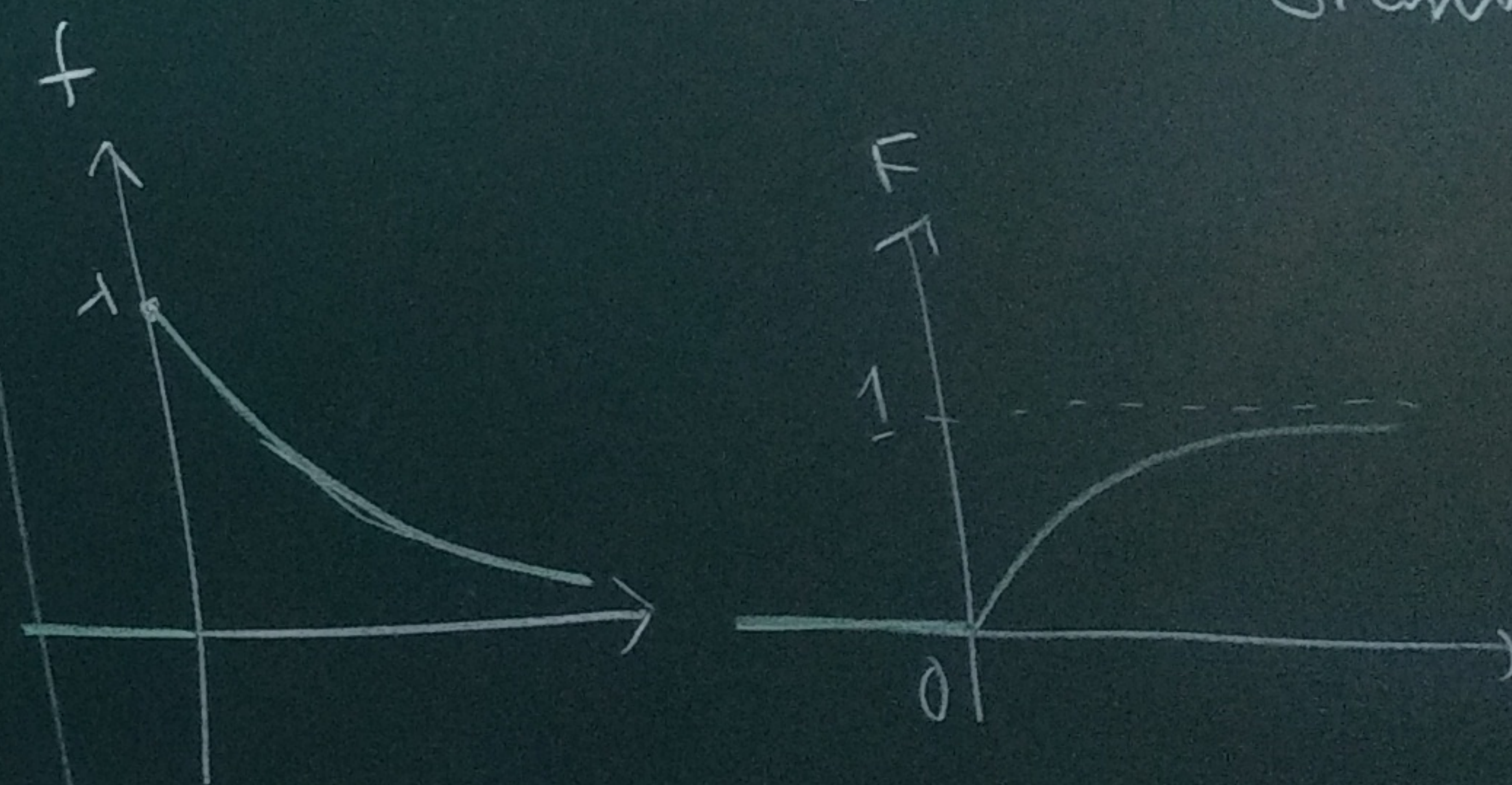
$$\lambda > 0$$

$$X \sim \text{Exponential}(\lambda)$$

let

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



Step 1: F

$$F_Y(y) = P$$

$$F_Y(y) =$$

$$(\because 1 - e^{-\lambda x} \leq$$

$$0 < y < 1$$

Let $Y = F(X)$

Distribution of Y ?

Step 1: Find distribution function of Y .

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F(X) \leq y) \\ &= P(1 - e^{-\lambda X} \leq y) \end{aligned} \quad \left. \vphantom{\begin{aligned} F_Y(y) &= P(Y \leq y) = P(F(X) \leq y) \\ &= P(1 - e^{-\lambda X} \leq y) \end{aligned}} \right\} \begin{aligned} P(F(X) \leq y) &= P(F(X) \leq y, X > 0) \\ &= P(1 - e^{-\lambda X} \leq y) \end{aligned}$$

$$F_Y(y) = 1 \quad y \geq 1, \quad F_Y(y) = 0 \quad y \leq 0$$

($\because 1 - e^{-\lambda X} \leq 1$) ($\because 0 \leq 1 - e^{-\lambda X}$)

$0 < y < 1$

$$\begin{aligned} P(Y \leq y) &= P(1 - e^{-\lambda X} \leq y) = P(-e^{-\lambda X} \leq y - 1) = P(e^{-\lambda X} \geq 1 - y) = P(-\lambda X \geq \ln(1 - y)) \\ &= P(X \leq -\frac{\ln(1 - y)}{\lambda}) = F(-\frac{\ln(1 - y)}{\lambda}) = 1 - e^{-\lambda(-\frac{\ln(1 - y)}{\lambda})} = y \end{aligned}$$

$$\therefore F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

step 2: $F_Y(\cdot)$ is piecewise differentiable

So $f_Y(y) = F'_Y(y)$ exists on $(-\infty, 0)$

$(0, 1)$

$(1, \infty)$

$$\text{Set } f_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 & 0 < y < 1 \\ 0 & y \geq 1 \end{cases}$$

observe that

$$P(a < Y \leq b) = P(Y \leq b) - P(Y \leq a)$$

$$= F_Y(b) - F_Y(a)$$

$$= \int_a^b f_Y(y) dy - \int_{-\infty}^a f_Y(y) dy$$

$$= \int_a^b f_Y(y) dy$$

$f_Y(\cdot)$ is the p.d.f of Y

consequently $Y \sim \text{Uniform}(0,1)$ $\square$

Example 5.2.8 (Contd)

(2) $Y \sim \text{Uniform}(0,1)$

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$g: (0,1) \rightarrow \mathbb{R}$$

$$g(y) = F^{-1}(y) \\ = -\frac{\ln(1-y)}{\lambda}$$

(Want to call it x but just to be precise)

$$W = g(Y)$$

Step 1: Distribution function of W

$$F_W(w) = P(W \leq w) = P(g(Y) \leq w) = P(g(Y) \leq w, 0 < Y < 1)$$

$$= P\left(\frac{-\ln(1-Y)}{\lambda} \leq w, 0 < Y < 1\right)$$

$$= P\left(Y \leq 1 - e^{-\lambda w}, 0 < Y < 1\right)$$

$$P(0 < Y < 1) = 1 = P(Y \leq 1 - e^{-\lambda w}) = F_Y(1 - e^{-\lambda w})$$

$$w \leq 0 \Rightarrow 1 - e^{-\lambda w} \leq 0$$

$$w > 0 \Rightarrow 0 < 1 - e^{-\lambda w} < 1$$

$$F_W(w) = \begin{cases} 0 & w \leq 0 \\ 1 - e^{-\lambda w} & w > 0 \end{cases}$$

$$F_Y(y) = y \quad 0 \leq y < 1$$

step 2: $F_W(\cdot)$ is piecewise differentiable

$$f_W(w) = F'_W(w) = \begin{cases} 0 & w \leq 0 \\ \lambda e^{-\lambda w} & w > 0 \end{cases}$$

again as done previously, $P(a < W \leq b) = \int_a^b f_W(w) dw$

obv. $f_W(\cdot)$ is p.d.f. of W

$\Rightarrow W \sim \text{Exponential}(\lambda)$

Possible
General "Idea" :-

• Want to create dataset where Probability density function
is $f: \mathbb{R} \rightarrow \mathbb{R}$

Step 1. Generate random number in $(0,1) \equiv Y$

Step 2. let $F(y) = \int_{-\infty}^y f(x) dx \quad y \in \mathbb{R}$

Step 3. let $X = F^{-1}(Y)$

then X has p.d.f

f

• Need $\begin{cases} F - \text{continuous} \\ F - \text{piecewise diff} \\ F' \text{ exists} \end{cases}$

Example 5

• Suppose we have $y =$

• choose

• Draw a

$(0,1)$ to

angle θ

line $y =$

What

$$= F'(Y)$$

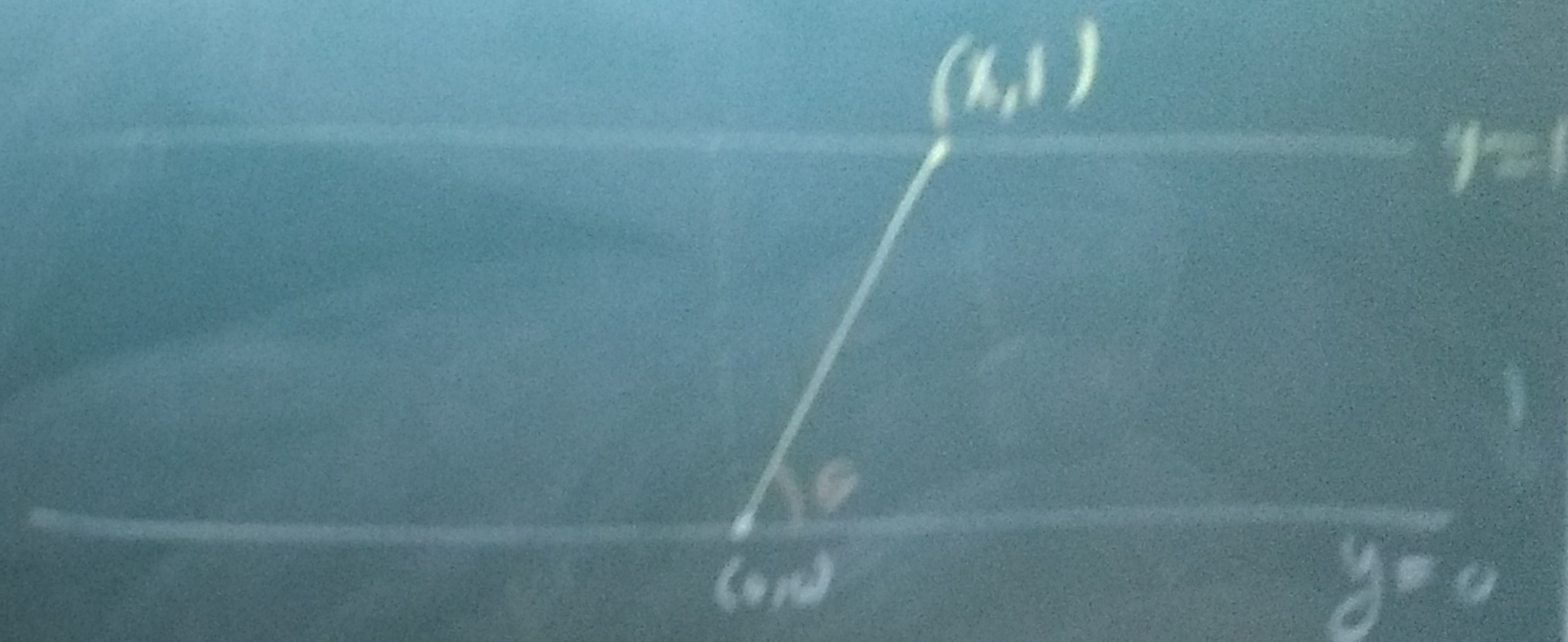
p.d.f

left

Example 5.2.9

- Suppose we are standing on the line $y=0$ at $(0,0)$
- choose $\Theta \sim \text{Uniform}(0, \pi)$
- Draw a line segment from $(0,0)$ to line $y=1$ at angle Θ & let it touch line $y=1$ at $(X, 1)$

What is distribution of X ?



observe: $\phi = \frac{\pi}{2} - \theta$

$X = \tan(\phi)$

$\theta \sim \text{Uniform}(0, \pi)$; θ has pdf given by

$$f_{\theta}(\theta) = \begin{cases} \frac{1}{\pi} & 0 < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

$\phi = \frac{\pi}{2} - \theta$

$E_{X|}$ $\phi \sim \text{Uniform}(-\pi/2, \pi/2)$

$f_{\phi}(\phi)$

in p

step 1

$F_X(x) =$

$$f_{\phi}(x) = \begin{cases} \frac{1}{\pi} & -\pi/2 < \phi < \pi/2 \\ 0 & \text{otherwise} \end{cases}$$

is p.d.f of ϕ .

Step 1: Find distribution function of X .

$$F_X(x) = P(X \leq x) = P(\tan(\phi) \leq x)$$

$$= P(\tan(\phi) \leq x, -\pi/2 < \phi < \pi/2)$$

$$= P(\phi \leq \arctan(x), -\pi/2 < \phi < \pi/2)$$

$$= \int_{-\pi/2}^{\arctan(x)} \frac{1}{\pi} d\phi$$

$$= \frac{1}{\pi} [\arctan(x) + \pi/2]$$

Step 2: $F(x)$ is differentiable

$$\& f(x) = F'(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

(Cauchy)
 X is a random variable
 with p.d.f $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

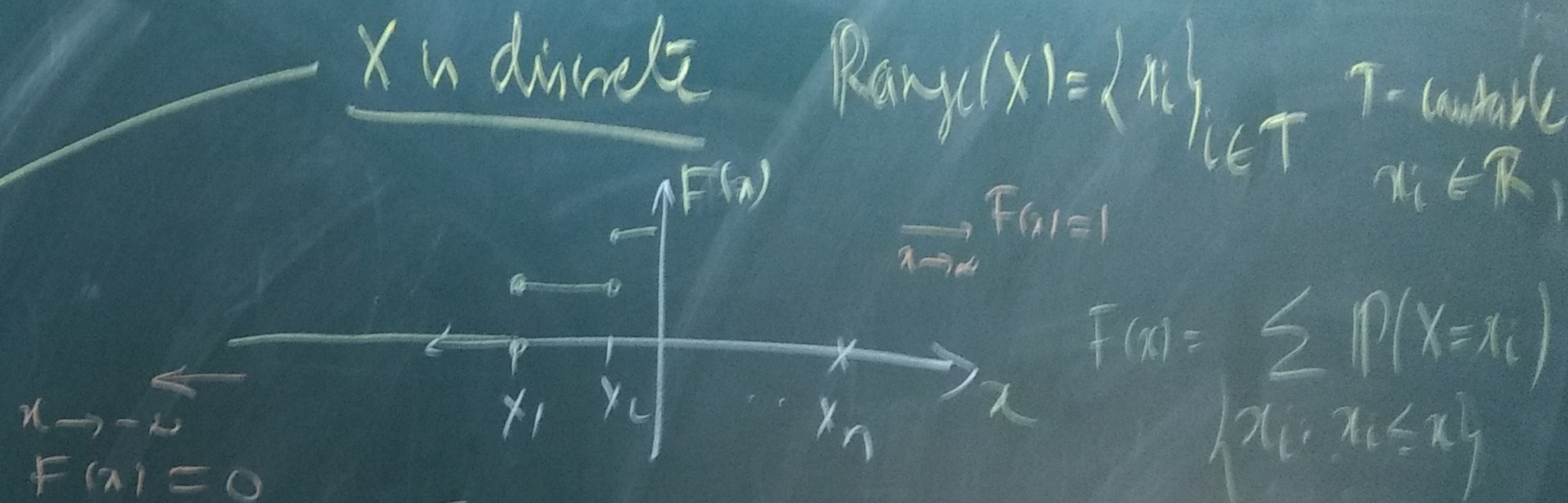
X = random variable

$$F(x) = P(X \leq x)$$

X - random variable

F - distribution function of X .

$$F(x) = P(X \leq x)$$

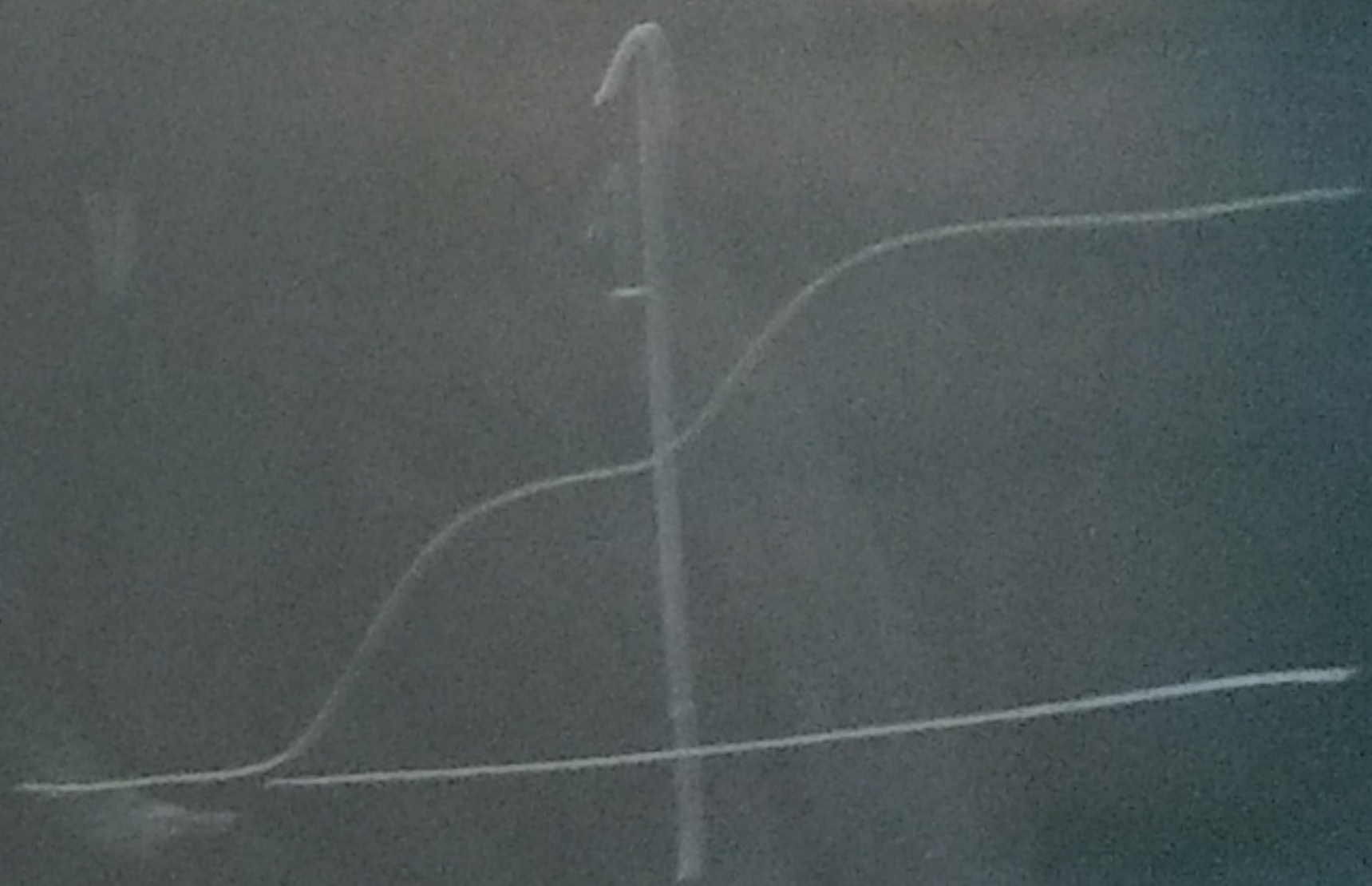


Right continuous Σ [discontinuous at x_i , jumps by $P(X=x_i)$]

X is continuous $P(X \leq x) = \int_{-\infty}^x f_X(y) dy$ $f_X(\cdot)$ pdf of X

$F_X(\cdot)$ is continuous, increasing

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow \infty} F_X(x) = 1$$



Properties of a distribution function

X be a continuous random variable (with p.d.f f)

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(y) dy \quad F: \mathbb{R} \rightarrow [0, 1]$$

(1) $F(\cdot)$ is continuous [Calculus]

(2) $F(\cdot)$ is monotonically increasing [$f \geq 0$]

(3) $F(\cdot)$ is piecewise differentiable ($F' = f$)

(4) $\lim_{x \rightarrow \infty} F(x) = 1$ $\lim_{x \rightarrow -\infty} F(x) = 0$ [Calculus / Probabilities]

X is discrete random variable. $\text{Range}(X) = \{x_i \mid i \in T\}$, T is countable. $F(x) = P(X \leq x)$

① F is monotonically increasing

② F is right-continuous $F(x+) = F(x)$.
jump-discontinuous at $x_i \in \text{Range}(X)$ $\xrightarrow{x_i} \overbrace{P(X=x_i)}$

③ $\lim_{x \rightarrow \infty} F(x) = 1$ $\lim_{x \rightarrow -\infty} F(x) = 0$

Lemma 5.3.7 $U \sim \text{Uniform}(0,1)$ X - continuous random variable such that its distribution function $F_X(\cdot)$ is **STRICTLY** increasing continuous function.

Then (a) $Y = F_X^{-1}(U)$ has the same distribution as X .

(b) $Z = F_X(X)$ then Z has the same distribution as U .

Proof :- (a) $P(Y \leq y) = P(F_X^{-1}(U) \leq y) = P(U \leq F_X(y))$

$$\begin{aligned} P(a < X \leq b) &= F_X(b) - F_X(a) \\ &= F_Y(b) - F_Y(a) \\ &= P(a < Y \leq b) \end{aligned}$$

$$= F_X(y)$$

The distribution function of X and Y are the same. Hence $X \sim Y$.

(b)

$$z \leq 0, \quad P(Z \leq z) = P(F_X(X) \leq z) = 0$$

$$P(Z \leq z) = P(F_X(X) \leq z) = 1$$

$$P(Z \leq z) = P(F_X(X) \leq z) = P(X \leq F_X^{-1}(z)) = F_X(F_X^{-1}(z)) = z$$

$$Z \sim U$$

$$0 < z < 1$$

$X \sim$ continuous random variable with pdf $f_X(\cdot)$
 $a \neq 0$ $a \in \mathbb{R}$ $b \in \mathbb{R}$

$$Y = aX + b$$

$$P(Y \leq y) = P(aX + b \leq y)$$

$$= P(aX \leq y - b)$$

$a > 0$

$$P(X \leq y - \frac{b}{a})$$

$a < 0$

$$P(X \geq y - \frac{b}{a})$$

$$P(Y \leq y) = \begin{cases}$$

$\Rightarrow$

F_Y

$=$

$$P(Y \leq y) = \begin{cases} \int_{-\infty}^y f_X(z) dz & a > 0 \\ \int_{\frac{y-b}{a}}^{\infty} f_X(z) dz & a < 0 \end{cases}$$

$\Rightarrow F_Y(\cdot)$ is piecewise differentiable

$$f_Y(y) = F'_Y(y) = \begin{cases} f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a} & a > 0 \\ f_X\left(\frac{y-b}{a}\right) \left(-\frac{1}{a}\right) & a < 0 \end{cases}$$

Y is also continuous random variable & its pdf is given by

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

A word about individual outcomes for continuous random variables

X is continuous random variable $P(X=x) = 0$ for all $x \in \mathbb{R}$

Let $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ & $g(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

f & g are different functions, $f(0) \neq g(0)$ & $f(1) \neq g(1)$

However, A - Borel set in $\mathbb{R}$

$(f \text{ & } g \equiv \text{same Probability})$ $Q(A) = \int_A f(x) dx$

$$P(A) = \int_A g(x) dx$$

$$\left\{ \begin{array}{l} Q = P \end{array} \right.$$

f & g are densities (i.e. $f \geq 0$, f is p.c., $\int_{-\infty}^{\infty} f(x) dx = 1$
 $g \geq 0$, g is p.c., $\int_{-\infty}^{\infty} g(x) dx = 1$)

& $\{x \in \mathbb{R} \mid f(x) \neq g(x)\}$ is countable

then " f & g " generate the same probabilities

We will regard f & g as the "same" density.

$$\int_{-\infty}^z f(y) dy = \int_{-\infty}^z g(y) dy \text{ for all } z \in \mathbb{R}$$

Pa

Theorem

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events

$$\textcircled{1} \quad A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

$$A = \bigcup_{n=1}^{\infty} A_n \text{ be an event}$$

$$P(A) = \lim_{n \rightarrow \infty} P(A_n)$$

$$\textcircled{2} \quad A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$

$$A = \bigcap_{n=1}^{\infty} A_n$$

$$\text{Then } P(A) = \lim_{n \rightarrow \infty} P(A_n)$$

$$\text{Proof: } B_1 = A_1 \quad B_2 = A_2 \cap A_1^c \quad B_1 \cup B_2 = A_2$$

$$B_n = A_n \cap A_{n-1}^c \quad \forall n \geq 2$$

$$\bigcup_{k=1}^n B_k = A_n$$

$$P(A_n) = P\left(\bigcup_{k=1}^n B_k\right)$$

$$B_k \cap B_l = \emptyset \quad k \neq l$$

$$= \sum_{k=1}^n P(B_k)$$

$$P\left(\bigcup_{k=1}^n B_k\right) = \sum_{k=1}^n P(B_k)$$

$$\lim_{n \rightarrow \infty} P(A_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n P(B_k) = P\left(\bigcup_{k=1}^{\infty} B_k\right) = P(A)$$

(2) Ex (Hint: $D = A_i^c$ apply ①) D

$$\forall \lim_{x \rightarrow \infty} F(x) = L$$

Ex $\left\{ \begin{array}{l} \varepsilon > 0 \text{ bc given} \\ \exists M \in \mathbb{N}(\varepsilon) \end{array} \right.$

$$x > M \Rightarrow |f(x) - L| < \varepsilon$$

$\boxed{(*) \lim_{n \rightarrow \infty} F(n) = 1}$ $F(n) = P(X \leq n)$

$$A_n = \{X \leq n\} \Rightarrow A_n \subseteq A_{n+1}$$

$\boxed{(*)}$

$$P(A_n) = F(n) \quad 8$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \{X \leq n\} = \{X \in \mathbb{R}\} = S$$

$$B_n = A_n$$

$$\Rightarrow \lim_{n \rightarrow \infty} P(A_n) = P(S) = 1$$

$$\lim_{n \rightarrow \infty} F(n)$$

(*)

$$\lim_{n \rightarrow -\infty} F(n) \parallel$$

$$\lim_{m \rightarrow \infty} F(-m)$$

$$\text{Ex } \lim_{x \rightarrow x} F(x) = a$$

$$A_m = \{X \leq m\}$$

$$\Rightarrow A_m \supseteq A_{m+1}$$

$$\bigcap_{m=1}^{\infty} A_m = \phi$$

$$\Rightarrow \lim_{m \rightarrow \infty} P(A_m) = 0$$