

Recall

$$X \sim \text{Normal}(\mu, \sigma^2)$$

$$\mu \in \mathbb{R} \quad \sigma \neq 0 \quad \sigma > 0$$

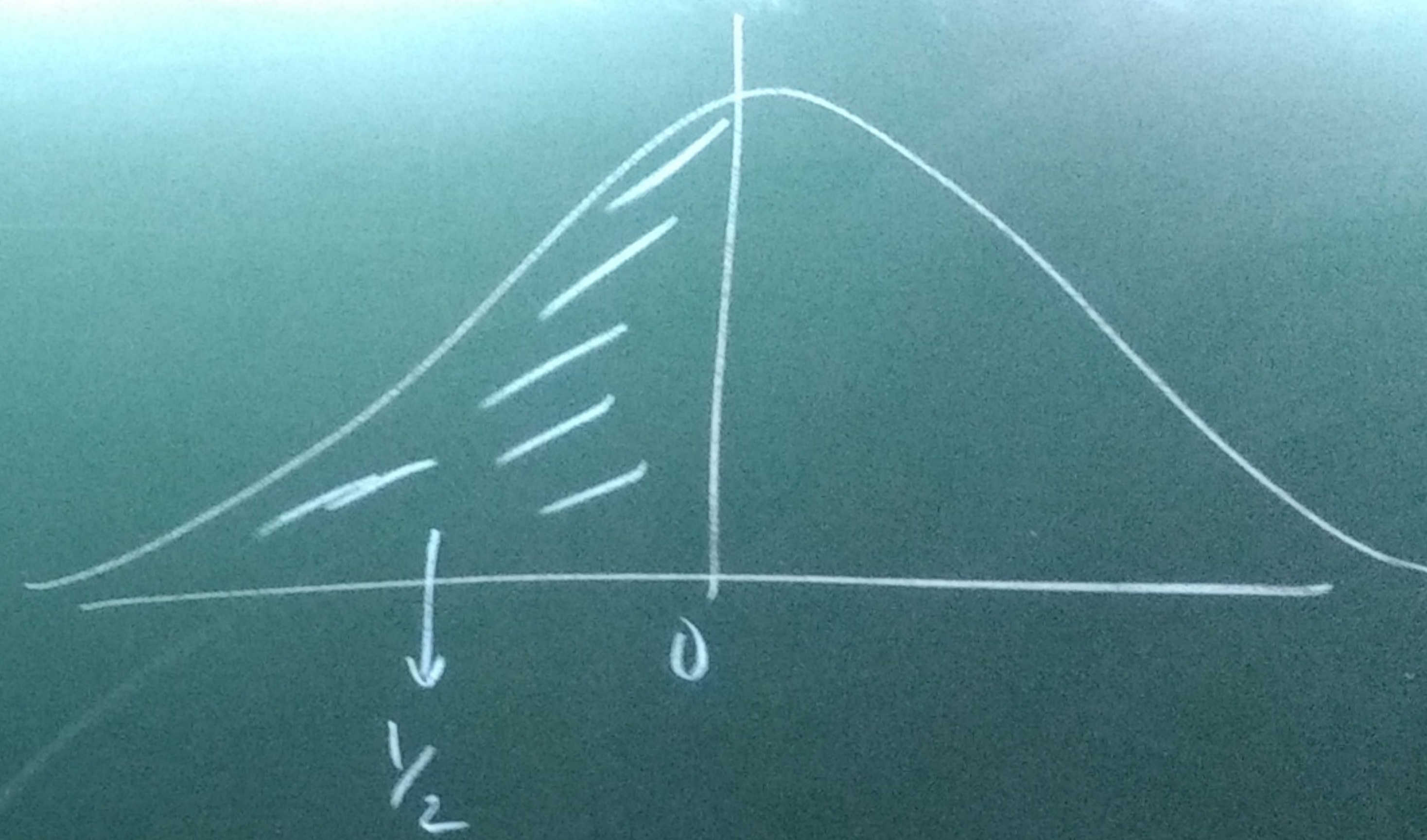
$$f_X(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi} \sigma} \quad x \in \mathbb{R}$$

$$\mathbb{P}(a < X \leq b) = ? \quad a, b \in \mathbb{R}$$

$$Z = \frac{X - \mu}{\sigma} \quad \text{then} \quad Z \sim \text{Normal}(0, 1)$$



$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^z \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$



$$f_z(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

- Symmetric i.e.  $f(z) = f(-z)$

$$\int_{-\infty}^{\infty} f_z(z) dz = 1$$

also

$$\int_{-\infty}^{\infty} f_z(z) dz = \int_{-\infty}^0 f_z(z) dz + \int_0^{\infty} f_z(z) dz$$

$$\stackrel{(z=-u)}{=} \int_0^{\infty} f_z(-u) du + \int_0^{\infty} f_z(z) dz$$

Symmetry

$$= 2 \int_0^{\infty} f_z(z) dz = 2\Phi(0)$$



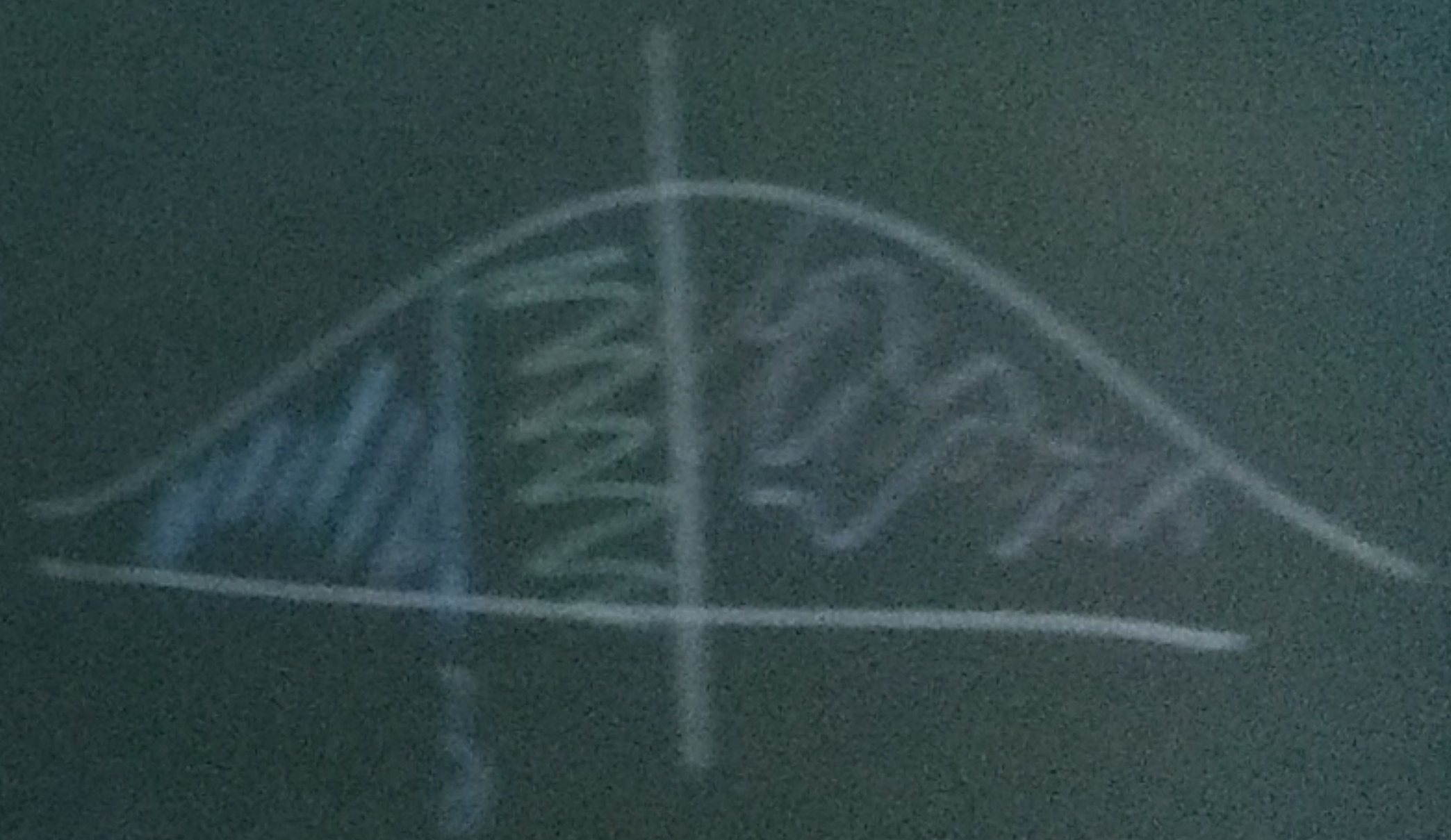
$$\Phi(0) = \frac{1}{2}.$$

$z > 0$

$$\begin{aligned}\Phi(z) &= \int_{-\infty}^z f_2(y) dy = \int_{-\infty}^0 f_2(y) dy + \int_0^z \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ &= \Phi(0) + \int_0^z \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ &= \frac{1}{2} + \int_0^z \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy\end{aligned}$$

$z < 0$

$$\begin{aligned}\Phi(z) &= \int_{-\infty}^z f_2(y) dy = \int_{-\infty}^0 f_2(y) dy - \int_0^z f_2(y) dy \\ &= 1 - \int_0^0 f_2(y) dy - \int_0^z f_2(y) dy\end{aligned}$$





(change of variable  $u = -y$ )

$$= 1 - \int_0^{-z} f_2(u) du - \int_{-\infty}^0 f_2(u) du$$

$$= 1 - \int_0^{-z} f_2(u) du - \Phi(0)$$

( $\Phi(0) = \frac{1}{2}$ )

( $-z > 0$ )  $\frac{1}{2} - \int_0^{-z} e^{-\frac{u^2}{2}} du$

Numerically computed table called Normal tables  
(Generate them in R as well)

$\int_0^a \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$  is calculated / tabulated for  $a > 0$

Summary

$z > 0$

$z < 0$

$a, b \in \mathbb{R}$   $a < b$

$P(a < X < b)$

$Z = \frac{X - \mu}{\sigma} \sim \text{Standard Normal}$



Standardizing

$$\Phi(0) = \frac{1}{2}$$

$$z > 0 \quad \Phi(z) = \frac{1}{2} + \int_0^z \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du$$

$$z < 0 \quad \Phi(z) = \frac{1}{2} - \int_0^{-z} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du$$

$a, b \in \mathbb{R} \quad a < b \quad X \sim \text{Normal}(\mu, \sigma^2)$

$$P(a < X < b) = P(a - \mu < X - \mu < b - \mu)$$

$$= P\left(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma}\right)$$

$$Z = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1)$$

$$= P\left(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma}\right)$$

$$= P\left(Z < \frac{b - \mu}{\sigma}\right) - P\left(Z < \frac{a - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right)$$

we know how to calculate

$$\Phi(z) \quad z \in \mathbb{R}$$

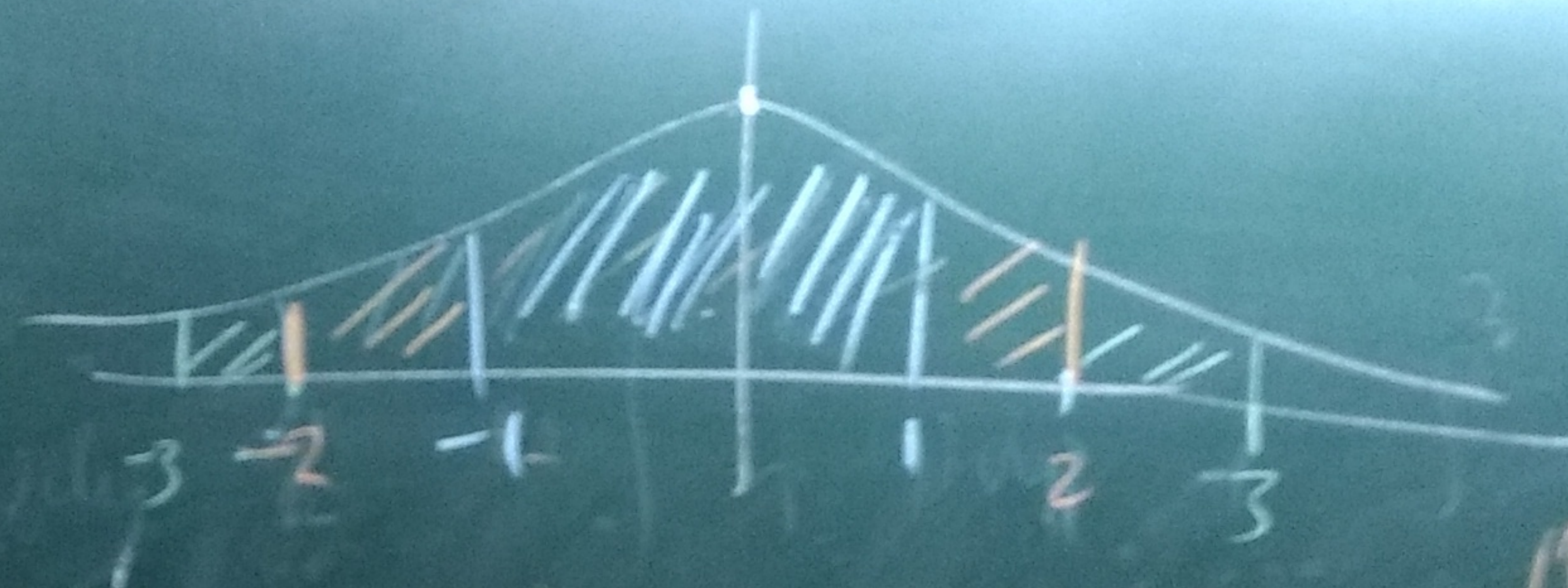
Remark

$$\frac{1}{2}\Phi(z) =$$



Remark

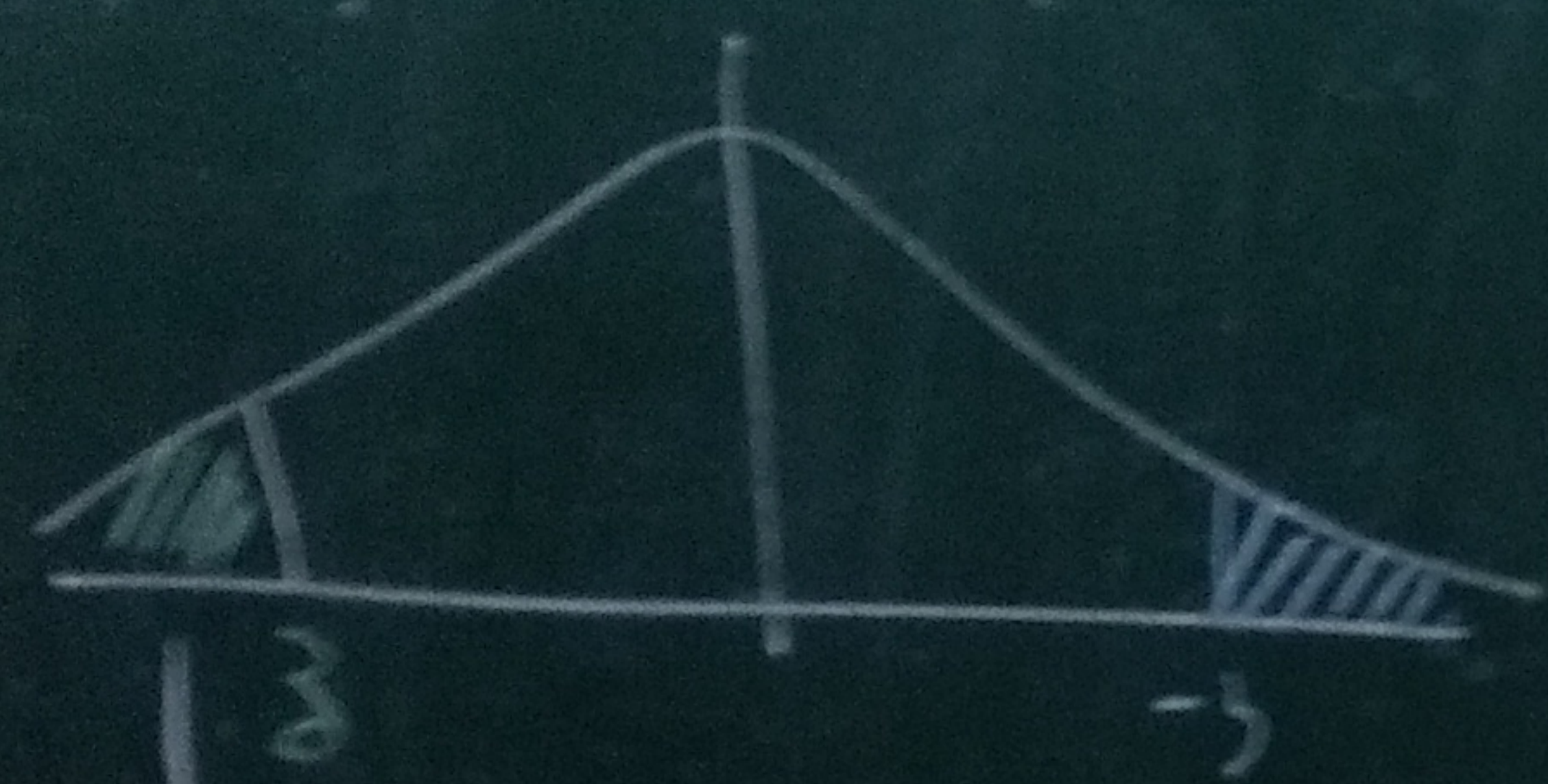
$$f_z(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$



$$P(-1 < Z < 1) \approx 0.68$$

$$P(-2 < Z < 2) \approx 0.95$$

$$P(-3 < Z < 3) \approx 0.997$$



$$\int_{-\infty}^z \frac{e^{-u^2/2}}{\sqrt{2\pi}} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-u^2/2} du$$



### 5.3 Transformation of Random Variables

$X$  - random variable  $g: \mathbb{R} \rightarrow \mathbb{R}$

$$Y = g(X)$$

Distribution of  $Y$ ?

Already known: (Discrete case)

$X$  is a discrete random variable

$\Rightarrow Y$  is a discrete random variable

and 
$$P(Y=y) = \sum_{\{x: g(x)=y\}} P(X=x)$$



Suppose  $X$  is a continuous Random Variable

$$Y = g(X)$$

Not so easy / No straight-forward formula  
is available

E.g.  $g(x) = 0 \quad \forall x \in \mathbb{R} \Rightarrow P(Y=0) = 1$

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad X \sim \text{Normal}(0,1)$$

$$Y = g(X) \Rightarrow P(Y=1) = P(X \geq 0) = 1 - \Phi(0) = \frac{1}{2}$$

$$\begin{aligned} P(Y=0) &= P(X < 0) \\ &= \Phi(0) \\ &= \frac{1}{2} \end{aligned}$$

$$\Rightarrow Y \sim \text{Bernoulli}(\frac{1}{2})$$

Example 5.3.1.

$X \sim \text{Uniform}(0,1)$

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$g(x) = x^2$$

$$Y = g(X)$$

Step 1.

Find

$$P(Y=y)$$

$$P(Y=y)$$

$$\sim$$

$$0 < y < 1$$

$$\sqrt{y} < x < 1$$

$$1 - \sqrt{y}$$



$$Y = g(X)$$

$$(Range(Y) = (0,1))$$

$$X \sim Uniform(0,1) \quad f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Step 1:-

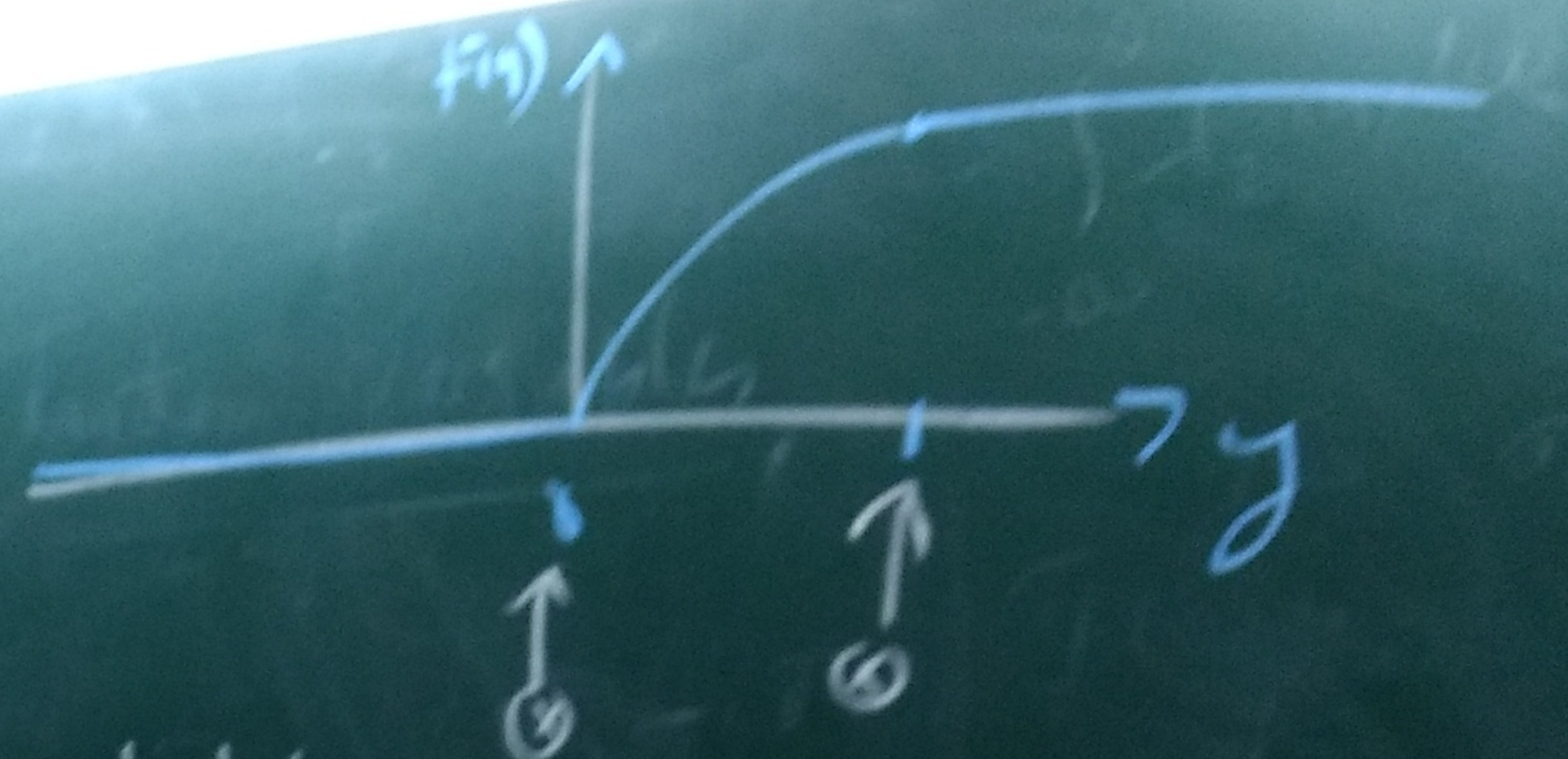
Find distribution function of  $Y$ .

$$P(Y \leq y) = P(X^2 \leq y) = \begin{cases} P(0 \leq X \leq \sqrt{y}) & y > 0 \\ 0 & y \leq 0 \end{cases}$$

$$\begin{cases} 0 & y \leq 0 \\ \sqrt{y} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases} = \begin{cases} \int_0^{\sqrt{y}} 1 dx & 0 < y < 1 \\ 0 & y < 0 \\ 1 & y \geq 1 \end{cases}$$



$$F(y) = P(Y \leq y)$$



$F$  is piecewise differentiable

$$F'(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & y > 1 \end{cases}$$

$$\begin{aligned} F(y) &= \int_{-\infty}^y f(z) dz \\ F'(y) &= f(y) \\ \text{where } f &= \text{pdf of } Y \end{aligned}$$

Fundamental theorem of calculus  $\Rightarrow$

$$f(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F(y) = \int_{-\infty}^y f(z) dz$$

variable

$$\Rightarrow P(Y \leq y) = \int_{-\infty}^y f(z) dz$$

$$= P(a < Y \leq b) = \int_a^b f(z) dz$$

$\Rightarrow Y$  is a continuous random variable with p.d.f

$$f: (R-) \rightarrow R \text{ given by } f(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$P(Y) \neq \text{Uniform}(0,1)$

as it assigns very  
large probabilities  
close to 0.

• p.d.f of  $Y$  is not  
a bounded function

Step 1:



$$Y = g(X) \quad (\text{Range}(Y) = (0,1))$$

$$X \sim \text{Uniform}(0,1) \quad P(f_X(x)) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Step 1:-

Find distribution function of  $Y$

$$P(Y \leq y) = P(X^2 \leq y) = \begin{cases} P(0 < X < \sqrt{y}) & y > 0 \\ 0 & y \leq 0 \end{cases}$$

$$\underbrace{\begin{cases} 0 & y \leq 0 \\ \sqrt{y} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}}_{P(Y \leq y)} = \begin{cases} \int_0^{\sqrt{y}} 1 dx & 0 < y < 1 \\ 0 & y < 0 \\ 1 & y \geq 1 \end{cases}$$