

Recall:

Expected value

weighted
average
of T

$X: S \rightarrow T$ $T \subseteq \mathbb{R}$, T - countable

$$E[X] = \sum_{t \in T} t P(X=t) = \sum_{t \in T} t f_X(t)$$

converges
absolutely

- $E[X]$ expectation of X exists & is finite

diverges to
 $\pm \infty$

- $E[X]$ is infinite

diverges

- $E[X]$ does not exist

lemma 4.16

$$E[X] < \infty$$

$\Leftrightarrow$

$$E[|X|] < \infty$$

$X: S \rightarrow T$ $Y: S \rightarrow U$ $T, U \subseteq \mathbb{R}$
Theorem 4.17 Let x, y be two discrete random variables

with both having finite expectations.

Let $a, b \in \mathbb{R}$.

$$(1) \quad E[ax] = a E[x]$$

$$(2) \quad E[x+y] = E[x] + E[y]$$

$$(3) \quad E[ax+by] = aE[x] + bE[y]$$

$$(4) \quad x \geq 0 \quad \text{then} \quad E[x] \geq 0.$$

$$(1, 2, 3 \subseteq (4))$$

Proof:-

$$E[ax]$$

$$Z = ax$$

(2)

$$Z$$

Theorem 4.17 $X: S \rightarrow T$ $Y: S \rightarrow U$ $T, U \subseteq \mathbb{R}$
Let $x \in Y$ be two discrete random variables

with both having finite expectations.

Let $a, b \in \mathbb{R}$.

(1) $E[ax] = a E[x]$

(2) $E[x+y] = E[x] + E[y]$

(3) $E[ax+by] = aE[x] + bE[y]$

(4) $x \geq 0$ then $E[x] \geq 0$.

(i.e. $T \subseteq [0, \infty)$)

Proof:-

$E[ax]$

$Z = ax$

(2) Z

Proof: -

① $a=0$ ✓

$X: S \rightarrow T$

$Z = aX$

$Z: S \rightarrow T'$

$T' = \{at \mid t \in T\}$
 $t=at$
 $for t \in T$

$$E[aX] = E[Z] = \sum_{s \in T'} s P(Z=s) = \sum_{t \in T} (at) P(X=t) \quad (a \neq 0)$$

$$= a \sum_{t \in T} t P(X=t) = a E[X]$$

(converges absolutely)

$Z = aX$ has finite expectation & $E[aX] = a E[X]$

②

$Z = X + Y$

$$E[Z] = \sum_{v \in V} v P(Z=v)$$

3

where
$$V = T + U = \{t + u \mid t \in T, u \in U\}$$

$$E[Z] = \sum_{t \in T} \sum_{u \in U} (t + u) P(X=t, Y=u)$$

$$= \sum_{t \in T} t \sum_{u \in U} P(X=t, Y=u)$$

Reordering of terms ✓ series converges absolutely

$$\sum_{u \in U} P(X=t, Y=u) = P(X=t)$$

$$+ \sum_{u \in U} u \sum_{t \in T} P(X=t, Y=u)$$

$$= \sum_{t \in T} t P(X=t) + \sum_{u \in U} u P(Y=u) = E[X] + E[Y]$$

4

$(3) \quad Z = aX, \quad W = bY$ from (1) $E[Z] = aE[X] \Leftarrow$
discrete random variables $E[W] = bE[Y] \Leftarrow$

apply (2) for Z & W

$$E[Z+W] = E[Z] + E[W]$$

$$\Rightarrow E[ax+by] = aE[X] + bE[Y]$$

$(4) \quad X \geq 0 \Rightarrow T \in [0, \infty) \Rightarrow \sum_{t \in T} t P(X=t) \geq 0$ as $t \geq 0$ & $P(X=t) \geq 0$
 $(T_n \geq 0 \Leftrightarrow \lim_{n \rightarrow \infty} T_n \geq 0)$
 $\Rightarrow E[X] \geq 0$

Examp

Method 1
let

Theorem

Method 2

Example 4.1.8

We roll a pair of dice. What is average of the sum of the outcomes on the top face?

Method 1 :-

Let X_1 be the outcome of dice 1

$X_1 \sim \text{Uniform}\{1, 2, \dots, 6\}$

X_2 be the outcome of dice 2

$X_2 \sim \text{Uniform}\{1, 2, \dots, 6\}$

$$E(X_1) = \sum_{t=1}^6 t P(X_1 = t) = \sum_{t=1}^6 t \cdot \frac{1}{6} = \frac{6 \cdot 7}{2} \cdot \frac{1}{6} = \frac{7}{2} = 3.5, (i=1, 2)$$

Theorem 4.1.7 (2) :- $E(X_1 + X_2) = E(X_1) + E(X_2) = 7$

Method 2 :- $Z = X_1 + X_2$, Find distribution of Z & $\sum_{t=1}^n t P(Z=t) = \dots$ Max calculation

Products of two random variables

$$X \sim \text{Uniform}\{1, 2, 3\}, \quad Y = 4 - X$$

$$E[X] = \sum_{t=1}^3 t \cdot P(X=t)$$

$$= \frac{1}{3}(1+2+3) = \frac{6}{3}$$

$$\boxed{Z = XY}$$

$$\text{Range}(Y) = \{1, 2, 3\}$$

$$P(Y=1) = P(X=3) = \frac{1}{3}$$

$$P(Y=2) = P(X=2) = \frac{1}{3}$$

$$P(Y=3) = P(X=1) = \frac{1}{3}$$

$$Y \sim \text{Uniform}\{1, 2, 3\}$$

$$E(Y) = \frac{6}{3}$$

$$\textcircled{3} Z = XY = X(4-X) = 4X - X^2 \quad E[Z] = 4E[X] - E[X^2]$$

$$U = X^2 \quad \text{Range}(U) = \{1, 4, 9\}$$

$$P(U=1) = P(X^2=1) = P(X=1) = \frac{1}{3}$$

$$P(U=4) = P(X^2=4) = P(X=2) = \frac{1}{3} \quad P(X=-2)=0$$

$$\text{Similarly } P(U=9) = P(X=3) = \frac{1}{3}$$

$$E[X^2] = E[U] = 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} + 9 \cdot \frac{1}{3} = \frac{14}{3}$$

$$(\text{Theorem 4.7 (1)}) \quad E(4X) = 4E(X) = 4 \cdot \frac{4}{3} = \frac{16}{3} \quad \& \quad E(-X^2) = -E(X^2) = -\frac{14}{3}$$

Theorem 4.1

Theorem 4.11

Theorem 4.12

Theorem 4.13

Theorem 4.17 (2) $\Rightarrow E[Z] = E[4X - X^2] = E[4X] + E[-X^2]$

So $E[XY] = E[Z] = \frac{24}{3} - \frac{14}{3} = \frac{10}{3}$

$E[X] E[Y] = \frac{6}{3} \cdot \frac{6}{3} = \frac{36}{9} \neq \frac{10}{3}$

In general $E[XY] \neq E[X]E[Y]$

Theorem 4.10 Let $X: S \rightarrow T$, $Y: S \rightarrow U$ be two INDEPENDENT discrete random variables having finite expectations. Then $Z = XY$ is a discrete random variable having finite expectation & $E[XY] = (E[X])(E[Y])$

Proof

$$X: S \rightarrow T \quad Y: S \rightarrow U \quad Z: S \rightarrow W$$

$$W = \{w \in \mathbb{R} \mid w = ut \quad u \in U, t \in T\}$$

(W - countable $\Rightarrow Z$ is a discrete random variable)

$$E[XY] = E[Z] = \sum_{w \in W} w P(Z=w)$$

$$= \sum_{t \in T} \sum_{u \in U} ut P(X=t, Y=u)$$

X & Y are independent

$$\sum_{t \in T} \sum_{u \in U} t \cdot u \cdot P(X=t) \cdot P(Y=u)$$

$$= \left(\sum_{t \in T} t P(X=t) \right) \left(\sum_{u \in U} u P(Y=u) \right)$$

$$= E[X] E[Y]$$

□

Ex. Is converse true?

ie $E[XY] = E[X]E[Y] \Rightarrow X \text{ & } Y \text{ are independent?}$

Recall:

$$E[X] = \sum_{t \in T} t P(X=t)$$

$$\begin{aligned} - E[aX + bY] &= a E[X] + b E[Y] \\ a, b &\in \mathbb{R} \end{aligned}$$

$$X \text{ \& } Y \text{ are independent} \Rightarrow E[XY] = E[X] E[Y]$$

In general

$$E[XY] \neq E[X] E[Y]$$

Example

$$X = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

$$E[X] = 0$$

$$Y = X^2, \quad P(Y=1) = 1$$

$$E[Y] = 1$$

$$E[XY] = E[X^3] = 0 = E[X]E[Y]$$

$$\left. \begin{aligned} P(X^3=1) &= P(X=1) = \frac{1}{2} \\ P(X^3=-1) &= P(X=-1) = \frac{1}{2} \end{aligned} \right\} \Rightarrow E[X^3] = 0$$

but X & Y are not independent

In general $E[XY] = E[X]E[Y] \stackrel{(*)}{\Rightarrow} X \text{ \& \& } Y \text{ are independent}$

Example 4-1-13 :-

$$X \sim \text{Bernoulli}(p)$$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

Example 4-1-14

$$X \sim \text{Binomial}(n, p)$$

$$R_X(X) = \{0, 1, \dots, n\}$$

$$P(X=k) = nC_k p^k (1-p)^{n-k}$$

$$E[X] = \sum_{k=0}^n k \cdot nC_k p^k (1-p)^{n-k}$$

$$= 0 + \sum_{k=1}^n \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np (p+1-p)^{n-1}$$

$$E[X] = np$$

Method 2 let $x_i \sim \text{Bernoulli}(p)$ $1 \leq i \leq n$
Independent

$$Z = \sum_{i=1}^n x_i$$

• show: $Z \sim \text{Binomial}(n, p)$

Theorem 4.17 (2) & Induction

$$E[Z] = \sum_{i=1}^n E[x_i]$$

Example 4.1.13

$$= \sum_{i=1}^n p = np$$

$$x \sim Z \Rightarrow E[x] = E[Z] \Rightarrow E[x] = np$$

Example 4.1.15 $X \sim \text{geometric}(p)$

$$\text{Range}(X) = \mathbb{N}$$

$$P(X=k) = p(1-p)^{k-1}$$

$$\begin{array}{l} p=0 \quad E(X)=0 \\ p=1 \quad E(X)=1 \end{array} \quad \text{don't count.}$$

$$0 < p < 1$$

Find: $\sum_{k=1}^{\infty} k p(1-p)^{k-1}$

$$\begin{aligned} T_n &= \sum_{k=1}^n k p(1-p)^{k-1} = \sum_{k=1}^n k (1-p)^{k-1} (1 - (1-p)) \\ &= \sum_{k=1}^n k (1-p)^{k-1} - \sum_{k=1}^n k (1-p)^k = 1 + \sum_{k=2}^n (1-p)^{k-1} - n(1-p)^n \end{aligned}$$

$$T_n = 1 + \sum_{k=1}^n (1-p)^{k-1} - n(1-p)^n$$

$$\lim_{n \rightarrow \infty} T_n = 1 + \frac{1-p}{p} = \frac{1}{p}$$

$$E[X] = \frac{1}{p}$$

$$E[X] = \sum_{k=0}^{\infty} k P(X=k)$$

$$S_n = \sum_{k=0}^n k \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= 0 + \sum_{k=1}^n \frac{\lambda^k e^{-\lambda}}{(k-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{n-1} \frac{\lambda^m}{m!}$$

$$\lim_{n \rightarrow \infty} S_n = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E[X] = \lambda$$

Analysis

$$a_n \rightarrow a \Rightarrow \begin{cases} b_n = a_{n+1} & n \geq 2 \\ b_1 = 0 \end{cases} \quad \lim_{n \rightarrow \infty} b_n = a$$

$$\lim_{n \rightarrow \infty} n a^n = 0 \quad \forall \quad 0 < a < 1$$

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n (1-p)^{k-1} = \frac{1-p}{p} \quad 0 < p < 1$$

$$\lim_{n \rightarrow \infty} \sum_{m=0}^n \sum_{l=0}^m = e^{\lambda}$$

Example 4.116 $X \sim \text{Poisson}(\lambda) \quad \lambda > 0$

$$\text{Range}(X) = \{0\} \cup \mathbb{N} \quad P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Expected value of $f(x_1, \dots, x_n)$

Method I: $Z = f(x_1, \dots, x_n)$ Find - distribution of Z
Compute $E[Z]$

Example 4.1.18 (Repeat Example 3.3.1)

$X \sim \text{Unif} \{-2, -1, 0, 1, 2\}$

$f(x) = x^2$ $Y = f(X)$

$$E[X] = \frac{-2 + -1 + 0 + 1 + 2}{5} = 0$$

$$P(Y=4) = 2/5$$

$$P(Y=1) = 2/5$$

$$P(Y=0) = 1/5$$

$$E[Y] = E[f(x)] = 0 \cdot \frac{1}{5} + 4 \cdot \frac{2}{5} + 1 \cdot \frac{2}{5}$$

$$= 0 \cdot P(X=0) + 4 \cdot (P(X=2) + P(X=-2)) + 1 \cdot (P(X=1) + P(X=-1))$$

$$= 0 \cdot P(X=0) + 4 \cdot P(X=2 \cup X=-2) + 1 \cdot P(X=1 \cup X=-1)$$

$$= 0 \cdot P(X=0) + 4 \cdot P(X=2) + 4 \cdot P(X=-2) + 1 \cdot P(X=1) + 1 \cdot P(X=-1)$$

$$= f(0) \cdot P(X=0) + f(2) \cdot P(X=2) + f(-2) \cdot P(X=-2) + f(1) \cdot P(X=1) + f(-1) \cdot P(X=-1)$$

$$E[f(X)]$$

$$= \sum_{k \in \text{Range}(X)} f(k) \cdot P(X=k)$$

Lemma 4

Proof

E[f(X)]

Theorem 4.119

$X: S \rightarrow T$ be a discrete random variable

$$f: T \rightarrow \mathbb{R}, u \in \mathbb{R} \quad Y = f(X)$$

$$E[f(X)] = \sum_{t \in T} f(t) P(X=t) \equiv \begin{cases} E[f(X)] < \infty & \text{if series converges absolutely} \\ \pm \infty & \Rightarrow E[f(X)] \text{ is infinite} \\ \text{diverge} & \Rightarrow E[f(X)] \text{ does not exist} \end{cases}$$

Proof:

$$u \in \mathbb{R}, \quad \{f(X)=u\} = \bigcup_{\substack{t \in T \\ f(t)=u}} \{X=t\}$$

$$E[f(X)] = \sum_{u \in \mathbb{R}} u \cdot P(f(X)=u)$$

$$= \sum_{u \in U} P\left(\bigcup_{t \in T: f(t)=u} \{X=t\}\right)$$

$$= \sum_{u \in U} \sum_{t \in T: f(t)=u} P(X=t)$$

(Rearrangement
of series)

$$= \sum_{\substack{u \in U \\ t \in T \\ f(t)=u}} f(t) P(X=t)$$

$$= \sum_{t \in T} f(t) P(X=t)$$

□