

4 Summarising Discrete Random Variables.

$$X: S \rightarrow T$$

random variable.

$T \subseteq \mathbb{R}$ - discrete
(Countable)

Probability
mass
function

$$f_X(t) = \mathbb{P}(X=t) \quad t \in T$$

$$Y: S \rightarrow T_1 \quad T_1 \subseteq \mathbb{R} \text{ countable}$$

$$\phi(x, y) = \mathbb{P}(X=x, Y=y) \quad (\text{Joint distribution of } X \& Y)$$

Exam

Average
(Basic)

$\{x_1, \dots, x_n\}$
 $\{y_1, \dots, y_m\}$

Example 2.1.2 (a) What is average number of success in n trials?

Average
(Basic)

1, 2, 3, 4, 5, 6

$$\text{Average} = \frac{1+2+3+4+5+6}{6}$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

each number a weight " $\frac{1}{6}$ "

— Summing a weighted sum.

Sum of weights $\equiv 1$

Average with respect to the given weights

$\{x_1, \dots, x_n\}$ — numbers

$\{w_1, \dots, w_n\}$ — weights

weighted
average

$$\sum_{i=1}^n \frac{w_i x_i}{\sum_{i=1}^n w_i}$$

$$v_i = \frac{w_i}{\sum_{i=1}^n w_i}$$

$$\sum_{i=1}^n v_i x_i$$

4.1 Expected Value

$X: S \rightarrow T$ ($T \subseteq \mathbb{R}$ countable) be a discrete random variable.

Then the expected value (or average value) of X

written as $E[X]$ is given by

$$E[X] = \sum_{t \in T} t P(X=t)$$

provided the sum converges absolutely.

$$\text{(i.e. } \sum_{t \in T} |t| P(X=t) < \infty)$$

$\{a_n\}_{n=1}^{\infty}$ sequence of real numbers.

$$T_n = \sum_{k=1}^n |a_k|$$

If $\lim_{n \rightarrow \infty} T_n = T \in \mathbb{R}$

then we say " $\sum_{n=1}^{\infty} a_n$ "

converges absolutely

$$a_n = \frac{(-1)^n}{n} \quad |a_n| = \frac{1}{n}$$

Discrete Random Variables

In this case $T_n = \sum_{k=1}^n (-1)^k \cdot 2^k \cdot P(X=2^k) = \sum_{k=1}^n (-1)^k$

suppose
$$= \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

T_n diverges

$E[X]$ does not exist

$X \sim$

$\Rightarrow$

(not

$X:$

• X is a constant random variable $X: S \rightarrow \mathbb{R}$
 $X(\omega) = c \quad \forall \omega \in S$ $\Rightarrow P(X=c) = 1$
 for some $c \in \mathbb{R}$

$\Rightarrow$

$$E[X] = \sum_{x \in \text{Range}(X)} x P(X=x) = c P(X=c) = c \cdot 1 = c$$

(notation: $E[c] = c$)

• $X: S \rightarrow T \quad T \subseteq \mathbb{R} \quad |T| < \infty \Rightarrow E[X]$ always exists & is finite
 number of elements in T

Example Expected Value

$X: S \rightarrow T$

$$T = \{2, 4, 8, \dots\} = \{2^n : n \geq 1\}$$

$$a_n = P(X = 2^n) \quad n \geq 1$$

$$\text{approx } a_n = \frac{1}{2^n}$$

$$E[X] = \sum_{n=1}^{\infty} 2^n P(X=2^n)$$

$$T_n = \sum_{k=1}^n 2^k \cdot \frac{1}{2^k} = n \Rightarrow T_n \rightarrow \infty$$

$E[X]$ is infinite $\therefore E[X] = \infty$

$$T = \{(-1)^n 2^n : n \geq 1\} \quad a_n = P(X = (-1)^n 2^n) = \frac{1}{2^n}$$

$$a_n = \frac{3}{4^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \frac{3}{4} + \frac{3}{16} + \dots = \frac{3}{4} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{3}{4} \cdot \frac{4}{3} = 1$$

$$E[X] = \sum_{n=1}^{\infty} 2^n \cdot \frac{3}{4^n}$$

$$= 3 < \infty$$

$$T_n = \sum_{k=1}^n (-1)^{k+1} P(X=i^k) = \sum_{k=1}^n (-1)^{k+1}$$

$$\Rightarrow \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

T_n diverges

$E[X]$ does not exist

Lemma 4.1.6

$X: S \rightarrow T$ be a discrete random variable

$|X|: S \rightarrow U$ be the discrete random variable such that

$$|X|(\omega) = |X(\omega)|$$

$$E[X] < \infty \iff E[|X|] < \infty$$

Proof: - $U = \text{Range}(|X|) = \{|t| : t \in T\}$

$$u \in U \quad P(|X|=u) = \begin{cases} P(X=u \vee X=-u) & \text{if } u \in T, -u \in T \\ P(X=u) & \text{if } u \in T \text{ \& } -u \notin T \\ P(X=-u) & \text{if } u \notin T \text{ \& } -u \in T \end{cases}$$

$P(X=a) = 0$
 $\forall a \notin \text{Range}(X)$

$$\sum_u P(|X|=u)$$

$$\sum_u P(X=u \vee X=-u)$$

$$u \in T \text{ or } -u \in T$$

$$\sum_u P(X=u)$$

$$u \in T \text{ or } -u \notin T$$

$$\sum_u P(X=-u)$$

$$u \notin T \text{ or } -u \in T$$

$$\sum_u P(|X|=u) = \sum_u |u| P(X=u) + \sum_u |u| P(X=-u) \quad \forall u \in U$$

$$E(|X|) = \sum_{u \in U} u P(|X|=u) = \sum_{u \in U} u P(X=u) + \sum_{u \in U} u P(X=-u)$$

$$= \sum_{u \in U} |u| P(X=u) + \sum_{u \in U} |u| P(X=-u)$$

(by def $u \geq 0$)

$$u \in U \Rightarrow |u|=u \text{ for some } t \in T$$

$$= \sum_{t \in T} |t| P(X=t)$$

Remember with random walk if times ≥ 0

$$\bigcup_{u \in U} P(|X|=u) =$$

$$\bigcup_{u \in U} P(X=u \vee X=-u)$$

$$u \in T \text{ \& } -u \in T$$

$$\bigcup_{u \in U} P(X=u)$$

$$u \in T \text{ \& } -u \notin T$$

$$\bigcup_{u \in U} P(X=-u)$$

$$u \notin T \text{ \& } -u \in T$$

$$u P(|X|=u) = |u| P(X=u) + |-u| P(X=-u) \quad \forall u \in U$$

$$E(|X|) = \sum_{u \in U} u P(|X|=u) = \sum_{u \in U} u P(X=u) + u P(X=-u)$$

$$= \sum_{u \in U} |u| P(X=u) + |-u| P(X=-u)$$

(by def $u \geq 0$)

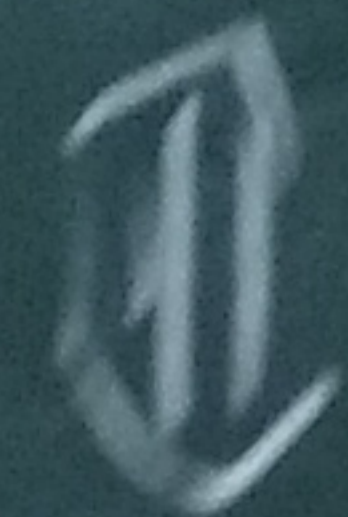
$$u \in U (=) |t|=u \text{ for some } t \in T$$

$$= \sum_{t \in T} |t| P(X=t)$$

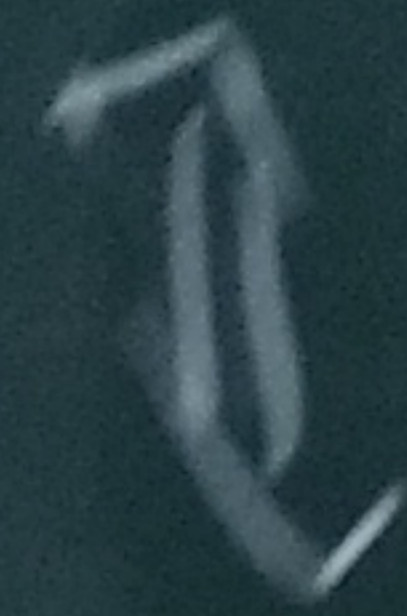
Re arrangement
possible only
if terms ≥ 0

$$E[|X|] = \sum_{t \in T} |t| P(X=t)$$

Suppose $E[|X|] < \infty$



$$\sum_{t \in T} |t| P(X=t) < \infty$$



$$\sum_{t \in T} t P(X=t) \text{ (converges absolutely)}$$

$$\Downarrow$$

$$E[X] < \infty$$