Due: Thursday, October 13th, 2016
Problem to be turned in: 2,4,13,14

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Sketch a graph of the function f.
   (b) Show that f is a probability density function.
   (c) Use f to calculate : $P((0, \frac{1}{4})$, $P((\frac{3}{2}, 2))$, $P((-3, -2))$ and $P((\frac{1}{2}, \frac{3}{2}))$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} k & \text{if } 0 < x < \frac{1}{4} \\ 2k & \text{if } \frac{1}{4} \le x < \frac{3}{4} \\ 3k & \text{if } \frac{3}{4} \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find k that makes f a probability density function.
   (b) Sketch a graph of the function f.
   (c) Use f to calculate : $P((0, \frac{1}{4})$, $P((\frac{1}{4}, \frac{3}{4}))$, $P((\frac{3}{4}, 1))$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} k \cdot \sin(x) & \text{if } 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determine the value of k that makes f a probability density function.
   (b) Calculate $P((0, \frac{1}{2}))$ and $P((\frac{1}{2}, 1))$.
   (c) Which will be larger, $P((0, \frac{1}{4}))$ or $P((\frac{1}{4}, \frac{1}{2}))$? Explain how you can answer this question without actually calculating either probability.
   (d) A game is played in the following way. A random variable X is selected with a density described by f above. You must select a number r and you win the game if the random variable results in an outcome in the interval $(r - 0.01, r + 0.01)$. Explain how you should choose r to maximize your chance of winning the game. (A formal proof requires only basic calculus, but it should take very little computation to determine the correct answer).

4. Let $\lambda > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } 0 < x \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that f is a probability density function.
   (b) Let $a > 0$. Find $P((a, \infty))$.

5. Let $a, b \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

   (a) Show that f is a probability density function.

(b) Show that if $I, J \subset [a,b]$ are two intervals that have the same length, then $P(I) = P(J)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{6}x^2 e^{-x} & \text{if } 0 < x \\ 0 & \text{otherwise} \end{cases}$$

Show that f is a probability density function.

7. Suppose X was continuous random variable with distribution function F. Express the following probabilities in terms of F:

   (a) $P(a < X \le b)$, where $-\infty < a < b < \infty$
   (b) $P(a < X < \infty)$ where $a \in \mathbb{R}$.
   (c) $P(\mid X - a \mid \ge b)$ where $a, b \in \mathbb{R}$.

8. Let $R > 0$ and $X \sim$ Uniform $[0, R]$. Let $Y = \min(X, \frac{R}{10})$. Find the distribution function of Y.

9. Let X be a random variable with distribution function given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 < x < \frac{1}{4} \\ \frac{x}{2} + \frac{1}{8} & \text{if } \frac{1}{4} \le x < \frac{3}{4} \\ 2x - 1 & \text{if } \frac{3}{4} \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

   (a) Sketch a graph of the function F.
   (b) Use F to calculate : $P([0, \frac{1}{4}))$, $P([\frac{1}{8}, \frac{3}{2}])$, $P((\frac{3}{4}, \frac{7}{8}])$.
   (c) Find the probabilty density function of X.

10. Let X be a random variable whose probability density function $f : \mathbb{R} \to [0, 1]$ is given by

$$f(x) = \begin{cases} kx^{k-1}e^{-x^k} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the distribution function of X for $k = 2$.
   (b) Find the distribution function of X for general k.

11. Let X be a random variable whose probability density function $f : \mathbb{R} \to [0, 1]$ is given by

$$f(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2} & \text{if } -R < x < R \\ 0 & \text{otherwise} \end{cases}$$

12. Let X be a random variable whose distribution function $F : \mathbb{R} \to [0, 1]$ is given by

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{2}{\pi}\arcsin(\sqrt{x}) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Find the probability density function of X. The distribution of X is called the standard arcsine law.

13. Let X be a random variable with density $f(x) = 2x$ for $0 < x < 1$ (and $f(x) = 0$ otherwise). Calculate the distribution function of X.

14. Let $X \sim$ Uniform$(\{1, 2, 3, 4, 5, 6\})$. Despite the fact this is a discrete random variable without a density, the distribution function $F_X(x)$ is still defined. Find a piecewise defined expression for $F_X(x)$.