Solution to final Exam Probability theory I

1. Range(X) = {1,2,3,4} Range(Y) = {1,2,3,4}

(a)

	X=1	X=2	X=3	X=4
Y=1	0	1/12	1/12	1/12
Y=2	1/12	0	1/12	1/12
Y=3	1/12	1/12	0	1/12
Y=4	1/12	1/12	1/12	0

$$\begin{cases} \mathbb{P}(X=i) = \frac{1}{4} \\ \mathbb{P}(Y=i \mid X=i) = 0 \\ \mathbb{P}(Y=j \mid X=i) = \frac{1}{3} \quad j \neq i \end{cases}$$

$\therefore \ \mathbb{P}(X=i, Y=j) = \frac{1}{12} \quad i \neq j$

(b) $\mathbb{P}(X=i) = \frac{1}{4} \quad \mathbb{P}(Y=i) = \frac{1}{4} \quad i \in \{1,2,3,4\}$

$\mathbb{P}(X=i, Y=i) = 0 \implies X$ & Y are NOT independent

(c) $E[Y|X=1] = \sum_{j=1}^{4} j\,\mathbb{P}(Y=j|X=1) = 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3} + \frac{4\cdot 1}{3} = \frac{1}{3}$

2. (a) $\mathbb{P}(Y \le y) = \mathbb{P}(-\ln(X) \le y) = \mathbb{P}(X \geq e^{-y})$

$\therefore \ y < 0 \qquad \mathbb{P}(Y \le y) = 0.$

$y \geq 0 \qquad \mathbb{P}(Y \le y) = 1 - e^{-y}$

$$\left(F_X(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases} \right)$$

Distribution function of Y is piecewise differentiable

$\therefore$ Y has a p.d.f given by

$$f_Y(y) = \begin{cases} e^{-y} & y \geq 0 \\ 0 & y < 0 \end{cases} \implies Y \sim \text{Exponential}(1)$$

(b) let $T_n = \ln(G_n)$

$= \ln\left(\prod_{i=1}^{n} X_i\right)^{1/n} = \frac{1}{n}\sum_{i=1}^{n} \ln(X_i)$

$Y_i = -\ln(X_i)$ are i.i.d Exponential(1) from (a)

$\implies E[Y_i] = 1$

$\therefore$ By weak law of large numbers

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}\left(\left| \frac{-\sum_{i=1}^{n} \ln(X_i)}{n} - 1 \right| > \varepsilon \right) = 0 \quad \text{—} (*)$$

~~[illegible]~~ let $\delta > 0$ be given.

Consider the event

$$\{ |G_n - \tfrac{1}{e}| \le \delta \} = \{ -e\delta \le eG_n - 1 \le e\delta \}$$

$$= \{ 1 - e\delta \le eG_n \le 1 + e\delta \}$$

Assume $e\delta < 1$

$$= \{ \ln(1-e\delta) \le \ln G_n + 1 \le \ln(1+e\delta) \}$$

(by fact about $\ln(\cdot)$)

$$= \{ -C_3 e\delta \le \ln G_n + 1 \le C_2 e\delta \}$$

$$\therefore \mathbb{P}(|G_n - \tfrac{1}{e}| \le \delta)$$

$$= \mathbb{P}(-C_3 e\delta \le \ln G_n + 1 \le C_2 e\delta \}$$

when $e\delta < 1$.

Take $\varepsilon = \max\{ C_3 e\delta, C_2 e\delta \}$ in (*) to conclude

(Why?) $\lim_{n\to\infty} \mathbb{P}(|G_n - \tfrac{1}{e}| > \delta) = 0$

when $e\delta < 1$. But large δ follows trivially.

$\therefore G_n \to \frac{1}{e}$ in probability as $n \to \infty$.

3. let X be a random variable such that

(a) $E[X] = \mu < \infty$ & $\mathrm{Var}[X] = \sigma^2 < \infty$. Then

$$\mathbb{P}(|X - \mu| \geq k\sigma) \le \frac{1}{k^2} \quad \forall k \geq 0$$

(b) let $X \equiv X_n \overset{d}{=} \text{Poisson}(n)$. For $\theta > 0$

Then $\mathbb{P}(X_n \le n\theta) = \sum_{0 \le j \le n\theta} \mathbb{P}(X_n = j)$

$= \sum_{0 \le j \le n\theta} \frac{e^{-n} n^j}{j!}$ (our quantity of interest)

$E(X_n) = n \quad \text{var}(X_n) = n$ [as $X_n \sim \text{Poisson}(n)$]

(a) $\Rightarrow$ $\therefore \mathbb{P}(|X_n - n| \ge k\sqrt{n}) \le \frac{1}{k^2} \quad \forall k > 0$ — (+)

let ~~θ<0~~. For $\theta > 0$

$\mathbb{P}(X_n \le n\theta) = \mathbb{P}(X_n - n \le n(\theta - 1))$

$= 1 - \mathbb{P}(X_n - n > n(\theta-1))$

let $\theta > 1$

$\mathbb{P}(X_n \le n\theta) = 1 - \mathbb{P}(X_n - n > n(\theta - 1))$

$\ge 1 - \mathbb{P}(|X_n - n| > n(\theta-1))$

Take $k = \sqrt{n}(\theta - 1)$ in (+) ($k \ge 0$)

$\ge 1 - \frac{1}{n(\theta-1)^2}$

$\therefore 1 - \frac{1}{n(\theta-1)^2} \le \mathbb{P}(X_n \le n\theta) \le 1 \quad \forall n \ge 1$

$\Rightarrow \lim_{n \to \infty} \sum_{j \le n\theta} \frac{e^{-n} n^j}{j!} = \lim_{n\to\infty} \mathbb{P}(X_n \le n\theta) = 1$

when $\theta > 1$

(b) $0 < \theta < 1$

$$\mathbb{P}(X_n - n \le n(1-\theta)) = \mathbb{P}\left(\frac{(X_n - n)}{\sqrt{n}} \le -\sqrt{n}(1-\theta)\right)$$

let $k = \frac{\sqrt{n}(1-\theta)}{2} \geq 0.$

$\therefore$ From (+)

$$1 - \frac{1}{k^2} \le \mathbb{P}\left(-k \le \frac{X_n - n}{\sqrt{n}} \le k\right) \le 1$$

Substituting for k

$$1 - \frac{4}{n(1-\theta)^2} \le \mathbb{P}\left(-\frac{\sqrt{n}(1-\theta)}{2} \le \frac{X_n - n}{\sqrt{n}} \le \frac{\sqrt{n}(1-\theta)}{2}\right) \le 1$$

$$\Rightarrow \lim_{n \to \infty} \mathbb{P}\left(\frac{|X_n - n|}{\sqrt{n}} \le \frac{\sqrt{n}(1-\theta)}{2}\right) = 1$$

(why?)

As $\mathbb{P}\left(\frac{(X_n - n)}{\sqrt{n}} \geq -\sqrt{n}(1-\theta)\right)$

$$\le 1 - \mathbb{P}\left(\frac{|X_n - n|}{\sqrt{n}} \le \frac{\sqrt{n}(1-\theta)}{2}\right)$$

$$\Rightarrow \mathbb{P}(X_n - n \le n(1-\theta)) \longrightarrow 0 \text{ as } n \to \infty$$

when $0 < \theta < 1$

$$\therefore \lim_{n \to \infty} \sum_{j \le n\theta} \frac{e^{-n} n^j}{j!} = 0$$

4. (a) $X \sim \text{Normal}(\mu, \sigma^2)$ $\quad Y = \frac{X-\mu}{\sigma}$

$$P(Y \le y) = P(X \le \sigma y + \mu)$$

$$= \int_{-\infty}^{\sigma y + \mu} f_X(x)\,dx$$

p.d.f. of Y (by fundamental thm of integral calculus)

$$f_Y(y) = \sigma f_X(\sigma y + \mu)$$

$$= \sigma \frac{e^{-\frac{(\sigma y + \mu - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}, \quad y \in \mathbb{R}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \quad y \in \mathbb{R}$$

(b) Need to calculate $P(X \ge 65.45)$ where $X \sim \text{Normal}(65, 2.5)$

$$P(X \ge 65.45) = P\left(\frac{X-65}{2.5} \ge \frac{0.45}{2.5}\right)$$

$Z \sim \text{Normal}(0,1)$ from (a)

$$= P(Z \ge 0.18)$$

$$= 1 - P(Z \le 0) - P(0 \le Z \le 0.18)$$

$$= 1 - \frac{1}{2} - 0.0714 = 0.4286$$

$\Rightarrow$ 42.86 % are taller than Steple.

5. $Y \sim \text{Exponential}(\lambda)$

(a) $M_Y(t) = E[e^{tY}] = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\,dy$

$$= \int_0^{\infty} e^{ty} \lambda e^{-\lambda y}\,dy$$

~~[illegible]~~ $\boxed{\lambda > t}$

$$M_Y(t) = \lambda \int_0^{\infty} e^{(t-\lambda)y}\,dy$$

$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)y}\Big|_0^{\infty}$$

$$= \frac{\lambda}{\lambda - t} \quad \text{as } t < \lambda$$

$t = \lambda$ $\quad M_Y(t) = \int_0^{\infty} \lambda\,dy = \infty$

$t > \lambda$ $\quad M_Y(t) = \int_0^{\infty} \lambda e^{(t-\lambda)y}\,dy = \frac{\lambda}{t-\lambda} e^{(t-\lambda)y}\Big|_0^{\infty}$

$$= \infty$$

(b) $M_Y(\cdot)$ is diff in a neighbourhood of 0

$\Rightarrow$ $E[Y^3] = M_Y'''(0)$

$E[Y^4] = M_Y^{(4)}(0)$

$$M_Y'(t) = \frac{+\lambda}{(\lambda-t)^2}$$

$$M_Y''(t) = \frac{+2\lambda}{(\lambda-t)^3}$$

$$M_Y^{(4)}(t) = \frac{24\lambda}{(\lambda-t)^5}$$

$$M_Y'''(t) = \frac{6\lambda}{(\lambda-t)^4}$$

$\Rightarrow E[Y^3] = 6\lambda^{-3}$ $\quad E[Y^4] = 24\lambda^{-4}$