Due: Tuesday August 30th, 2005[1]
Problem 2,3 (a)-(c),4

1. Suppose μ is a finitely additive set function defined on $\mathcal{A}$.
   (i) Then, for all $A, B \epsilon \mathcal{A}, \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.
   In particular, if $\mu(\Omega) < \infty$ (so that there is no problem with subtraction), then
   $$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$
   (ii) Can you write down a general formula for $\mu(A \cup B \cup C), A, B, C \epsilon \mathcal{A}$, (or, more generally, for $\mu\left(\cup_{i=1}^{n} A_i\right)$, when $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$) - under the assumption that $\mu(\Omega) < \infty$?

2. Let Ω be a countable set and $\mathcal{A} = 2^{\Omega}$ be the collection of all subsets of Ω. Then,
   (i) $\mathcal{A}$ is a σ-algebra;
   (ii) if $\mu : \mathcal{A} \to [0, \infty]$ is defined by $\mu(E) =$ 'number of elements in E', then μ is a measure, and is called **the counting measure on** Ω (since μ counts the number of elements in a set).
   (iii) If $\Omega = \{w_1, w_2, \ldots\}$ is an enumeration of Ω, and if Ω is infinite, let $A_n = \{w_n, w_{n+1}, \ldots.\}$; deduce that $A_n \downarrow \emptyset$ but
   $$\mu(\emptyset) = 0 \neq \infty = \lim_{n\to\infty} \mu(A_n).$$
   (iv) If μ is a general possibly infinite measure defined on an algebra $\mathcal{A}$ of subsets of any set Ω, and if $A, A_n \in \mathcal{A}, A_1 \supseteq A_2 \supseteq, \ldots, A = \cap_{n=1}^{\infty} A_n$, then show that $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ provided there exists some k so that $\mu(A_k) < \infty$.

3. Let $I = [0, 1]$. Let $I_1 = I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the open middle third interval of I. Next, let I_{21} and I_{22} be the two open middle third intervals of $I - I_1$. Let $I_2 = I_{21} \cup I_{22}$. For $j \geq 3$ and $k = 1, 2, 3 \ldots, 2^{j-1}$, let I_{jk} be the open middle third intervals of $I - \cup_{k=1}^{j-1} I_k$ and let $I_j = \cup_{k=1}^{2^{j-1}} I_{jk}$. Finally, let $C = I - \cup_{j=1}^{\infty} I_j$. C is called the cantor set.
   (a) Show that C is compact, nowheredense and totally disconnected.
   (b) Show that C is an uncountable closed set.
   (c) Show that $\lambda(C) = 0$, where λ is lebesgue measure on $[0, 1]$.
   (d) For $\alpha \in (0, 1)$, construct a set C_α similar to C by removing open α intervals. Calculate $\lambda(C_\alpha)$
   (e) Show that there exists a Borel set $A \subset [0, 1]$ such that $0 < \lambda(A \cap I) < \lambda(I)$ for every subinterval I of $[0, 1]$.

4. Suppose $(\Omega, \mathcal{B}, \mu)$ is a σ-finite measure space, and suppose $\mathcal{A}$ is an algebra of subsets of Ω such that $\mathcal{B} = \sigma(\mathcal{A})$. Show that if $B \epsilon \mathcal{B}$, and if $\varepsilon > 0$, then there exists $A \epsilon \mathcal{A}$ such that $\mu(A \Delta B) < \varepsilon$ - where $A \Delta B = (A - B) \cup (B - A)$. (Hint : First consider the case when μ is finite; in that case show that the collection of sets B in $\mathcal{B}$ for which the desired conclusion holds, is a monotone class containing $\mathcal{A}$.)

[1]Please give the assignments to Ms. Asha in the Statmath office.

Due: Thursday September 1st, 2005[2]
Problem 6,7,8

5. Let P be any probability measure on $(\mathbb{R}, \mathcal{B})$. Show that for any borel set B and any $\epsilon > 0$ there is an open set $U \supset B$ and a compact set $K \subset B$ such that $P(U \cap K^c) < \epsilon$.

6. Let P be any probability measure on $(\mathbb{R}^2, \mathcal{B}^2)$, where $\mathcal{B}^2$ is the Borel σ-algebra on $\mathbb{R}^2$. Show that $P_1(B) = P(B \times \mathbb{R})$ is a probability measure on $(\mathbb{R}, \mathcal{B})$.

7. Let $\mathcal{A}$ denote the algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ considered in Problem 6 Hw 1. Let $\Omega = \mathbb{Q}$ denote the set of rational numbers in $\mathbb{R}$, and let $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{Q}$. Consider the measure μ defined on $\mathcal{A}_0$ by $\mu(A) = 0$ or ∞ according as $A = \emptyset$ or $A \neq \emptyset$. Show that there exist more than one measure on $\sigma(\mathcal{A}_0)$ (in $\mathbb{Q}$) which agree with μ on $\mathcal{A}_0$.

8. Let $(\Omega$, $\mathcal{B}$, $\mu)$ be a σ-finite measure space, let $\mathcal{A}$ be any algebra of subsets of Ω such that $\mathcal{B}= \sigma(\mathcal{A}$), and let $\mu_0 = \mu|\mathcal{A}$.

   (i) Show that $\mu_0^* = \mu^*$ (as functions on 2^{Ω}) and that in fact

   $$\mu_0^*(E) = \mu^*(E) = \inf\{\mu(B) : B \epsilon \mathcal{B}, E \subseteq B\}$$

   for all subsets E of Ω.

   (ii) If $N \subseteq \Omega$ and $\mu^*(N) = 0$, show that $N \,\epsilon\, \mathcal{M}(\mu)$; we say that $(\Omega, \mathcal{M}(\mu), \mu^*)$ is a 'complete' measure space - meaning that if $N \subseteq M, M \epsilon \mathcal{M}(\mu)$ and $\mu^*(M) = 0$, then $N \,\epsilon\, \mathcal{M}(\mu)$; i.e., $\mathcal{M}(\mu)$ contains all μ^* - null sets.

   (iii) Show that $E \epsilon \mathcal{M}(\mu)$ if and only if there exist $B_0, B_1 \epsilon \mathcal{B}$ such that $B_0 \subseteq E \subseteq B_1$ and $\mu(B_1 - B_0) = 0$. (Hint : First assume $\mu(\Omega) < \infty$. (The general case easily follows from this by the assumed σ-finiteness.) Use (i) to lay hands on B_1. Define B_0 to be the B_1 you would have got for E'.)

   (iv) Make precise the statement that $(\Omega, \mathcal{M}(\mu), \mu^*)$ is 'the completion' of $(\Omega, \mathcal{B}, \mu)$.

[2]Please give the assignments to Ms. Asha in the Statmath office.