Due: Tuesday August 9th

Problems to be turned in: 2,3(b), 7,9,10

1. Let $\mathbb{N}$ be the set of natural numbers. Show that $Card(\mathcal{P}(\mathbb{N}) = \mathbb{R}$.
2. Suppose $\Omega \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies that $A \cap B^c \in \mathcal{F}$ show that $\mathcal{F}$ is an algebra.
3. Show the following: (a) If $\mathcal{B}$ is a σ-algebra, then $\mathcal{B}$ is an algebra as well as a monotone class.

   (b) If $\mathcal{B}$ is an algebra as well as a monotone class, then $\mathcal{B}$ is a σ-algebra.
4. Solve the following: (a) How many distinct algebras of subsets of Ω exist, if Ω is a three element set?

   (b) If $\mathcal{A}$ is a finite algebra of subsets of (a possibly infinite set) Ω, can you say something about the number of distinct sets in $\mathcal{A}$?
5. If $\mathcal{A}$ is an algebra of sets in Ω, and if $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, then there exist sequences $\{S_n\}_{n=1}^{\infty}, \{D_n\}_{n=1}^{\infty}$ of sets in $\mathcal{A}$ such that: (i) $S_1 \subset S_2 \subset S_3 \subset$ ....;
   (ii) $D_n \cap D_m = \emptyset$ if $n \neq m$; and
   (iii) $\cup_{k=1}^{n} E_k = \cup_{k=1}^{n} S_k = \cup_{k=1}^{n} D_k$, for $n = 1, 2, ...$

   Further, conditions (i) - (iii) uniquely determine the sets S_n and D_n, for all n.
6. Let $\Omega = \mathbb{R}$, let $\mathcal{S}$ denote the collection of intervals of the form (a,b], where $-\infty \leq a < b \leq \infty$ (where of course $(a, \infty]$ is to be interpreted as (a, ∞)). Show that a typical non-empty element of $\mathcal{A}(\mathcal{S})$, the algebra generated by $\mathcal{S}$, is of the form $\coprod_{k=1}^{n} I_k$, where $n = 1, 2, \ldots$. and $I_k \epsilon \mathcal{S}$ for $1 \leq k \leq n$.
7. Let $\mathcal{A}$ be an algebra of subsets of a set Ω . Let $\Omega_0 \in \mathcal{A}$. Define $\mathcal{A} \cap \Omega_0 = \{A \cap \Omega_0 : A \epsilon \mathcal{A}\}$. Show that $\mathcal{A} \cap \Omega_0$ is an algebra of subsets of Ω_0, and that $\mathcal{A} \cap \Omega_0$ is a σ-algebra (respectively, monotone class) if $\mathcal{A}$ is. (In the case when $\mathcal{A} \cap \Omega_0 \in \Omega$, it is more natural and customary to write $\mathcal{A}|_{\Omega_0}$ instead of $\mathcal{A} \cap \Omega_0$.)
8. Assume $\mathcal{A}$ is a σ-algebra on Ω. Let $B \subset \Omega$. Let $C = \mathcal{A} \cup \{B\}$. Show that $\sigma(C) = \sigma\{(B \cap U) \cup (B^c \cap V) : U, V \in \mathcal{A}\}$
9. Assume $\mathcal{A}$ is a σ-algebra on Ω. If $\{A_1, A_2, \ldots A_n\}$ form a partition of Ω, then describe $\sigma(\mathcal{A} \cup \{A_1, A_2, \ldots, A_n\})$.
10. Let $\mathcal{A}_1, \mathcal{A}_2$ be two σ-algebras on Ω. Show that $\sigma(\mathcal{A}_1 \cup \mathcal{A}_2\} = \sigma(\mathcal{C})$ where $\mathcal{C} = \{A_1 \cup A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$.