Due: Tuesday, August 22nd, 2006
Problems to be turned in: 4, 8, 12

1. If $x > -1$ then $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}$.
2. Let $x \in \mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers. Show that the following are equivalent:
   (a) $\forall \epsilon > 0$, there is an $N \equiv N_\epsilon \in \mathbb{N}$ such that $\mid x_n - x \mid < \epsilon$ for all $n \geq N$.
   (b) Let $C > 0$, $\forall \epsilon > 0$, there is an $M \equiv M_\epsilon \in \mathbb{N}$ such that $\mid x_n - x \mid \leq C\epsilon$ for all $n > M$.
3. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers $\mathbb{R}$ and suppose that $x_n \to x$.
   (a) Let $m, n \in \mathbb{N}$, show that $x_{m+n} \to x$ as $m \to \infty$.
   (b) Let $m, l \in \mathbb{N}$, $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) = \sum_{k=0}^{l} p_k x^k$, and $q : \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $q(x) = \sum_{k=0}^{m} q_k x^k$, with $p_k \in \mathbb{R}, q_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Show that if $r : \mathbb{R} \to \mathbb{R}$ defined by $r(x) = \frac{p(x)}{q(x)}$ then $r(x_n) \to r(x)$.
   (c) Show that $\{\mid x_n \mid\}_{n=1}^{\infty}$ also converges
4. Find: (i) $\lim_{n\to\infty} \frac{2^n}{n!}$, (ii) $\lim_{n\to\infty} \sqrt{n^2 - n} - n$ and, (iii) $\lim_{n\to\infty} a_n$, where $b \in (0,1)$ and $a_n = nb^n, n \in \mathbb{N}$.
5. Let $\alpha > 0$. Consider $\{x_n\}_{n=1}^{\infty}$, such that $x_n = \frac{n^\alpha}{(1+p)^n}$ for $n \in \mathbb{N}$. Decide if $\{x_n\}_{n=1}^{\infty}$converges or not.
6. Consider the $\{y_n\}_{n=1}^{\infty}$, such that $y_1 > 1$ and $y_{n+1} := 2 - \frac{1}{y_n}$ for $n \geq 2$. Show that y_n converges.
7. Consider $\{x_n\}_{n=1}^{\infty}$, such that $x_n = \left(1 + \frac{1}{n}\right)^n$, for all $n \in \mathbb{N}$. Show that x_n is a monotonically increasing sequence and it converges to $x \in \mathbb{R}$.
8. Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define $s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$ for $n \in \mathbb{N}$.
   (a) Show that s_n is monotonically decreasing and $\lim_{n\to} s_n = \sqrt{a}$.
   (b) If $z_n = x_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$.
   (c) Justify the statement: "this is a good algorithm for calculating square roots".
9. Let $\{z_n\}_{n=1}^{\infty}$be a sequence of real numbers such that $L := \lim_{n\to\infty} \frac{z_{n+1}}{z_n}$ exists. If $L < 1$, then $x_n \to 0$. What happens if $L > 1$?
10. Suppose $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$are such that for every $\epsilon > 0$ there is an M such that $\mid x_n - y_n \mid < \epsilon$ for all $n \geq M$. If $x_n \to x$ then does it imply that y_n converges.
11. Let $r < 1$ and $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers such that $x_n = \sum_{k=1}^{n} r^k$ for all $n \in \mathbb{N}$. Show that x_n is a convergent sequence.
12. Let A be a bounded non-empty subset of $\mathbb{R}$. Let $s = \sup(A)$ and $i = \inf(A)$. Show that there are sequences $\{x_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$in A such that $x_n \to s$ and $y_n \to i$ as $n \to \infty$.
13. Give Examples of the following:
    (a) a bounded sequence $\{z_n\}_{n=1}^{\infty}$that does not converge.
    (b) sequences $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$that do not converge but their sum converges
    (c) sequences $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$that do not converge but their product converges.