CLASSICAL DESCRIPTIVE SET THEORY

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Topological Preliminaries.

Definition. A metric d on a non-empty set X is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions. For every $x, y, z \in X$,

(1) $d(x, y) = 0 \Leftrightarrow x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \le d(x, y) + d(y, z)$.

Example. Let $X = \omega = \{0, 1, 2, \ldots\}$ be the set of all natural numbers and $d(m, n) = 1$ whenever $m \ne n$. Of course, $d(m, n) = 0$ whenever $m = n$.

Example. Let $X = \mathbb{R}^n$ or X a subset of $\mathbb{R}^n$ such as $[0, 1]^n$. For $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$, set

$$d((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = \sqrt{\sum_1^n (x_i - y_i)^2}.$$

Example. Let $X = \omega^\omega$, the set of all sequences of natural numbers, or $X = \{0, 1\}^\omega$, the set of all sequences of 0's and 1's. For $\alpha = \{\alpha(0), \alpha(1), \ldots\}, \beta = \{\beta(0), \beta(1), \ldots\}$ in X, $\alpha \ne \beta$, define

$$d(\alpha, \beta) = \frac{1}{\text{The least } n(\alpha(n) \ne \beta(n)) + 1}.$$

Exercise. Check that d defined in all the above examples is a metric.

Definition. Let (X, d) be a metric space, $x \in X$ and $r > 0$, set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

We call $B(x, r)$ the open ball in X with centre x and radius r.

Exercise. Let $B(x, r)$ and $B(y, s)$ be open balls in a metric space (X, d) and $z \in B(x, r) \cap B(y, s)$. Show that there is a positive real number t such that

$$B(z, t) \subset B(x, r) \cap B(y, s).$$

Definition. A sequence $\{x_n\}$ in a metric space is said to converge to a point $x \in X$ if

$$\forall \epsilon > 0 \exists N \forall n \geq N (d(x_n, x) < \epsilon).$$

A sequence $\{x_n\}$ is called a Cauchy sequence if

$$\forall \epsilon > 0 \exists N \forall m, n \geq N (d(x_m, x_n) < \epsilon).$$

Exercise. Show that every convergent sequence is Cauchy.

Example. Let $\mathbb{Q}$ be the set of all rational numbers. Set

$$x_n = \sum_{1}^{n} \frac{1}{i!}.$$

Show that $\{x_n\}$ is a Cauchy sequence that does not converge to any point in $\mathbb{Q}$.

Definition. A metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Exercise. Show that the metric spaces ω, ω^ω, $\{0,1\}^\omega$, $\mathbb{R}^n$ and $[0,1]^n$ defined in the beginning of this section are all complete.

Definition. Let (X, d) be a metric space. A subset U of X is open in X if

$$\forall x \in U \exists r > 0 (B(x,r) \subset U),$$

i.e., U is the union of a family of open balls.

Exercise. Show the following:

(1) $\emptyset$ and X are open in X.
(2) If U and V are open in X, so is $U \cap V$.
(3) If $\{U_\alpha\}$ is a family of open sets in X, so is $\cup U_\alpha$.
(4) There is a sequence of open sets in $\mathbb{R}$ whose intersection is not open.

(5) Every subset of ω is open in ω.

Definition. Let (X, d) be a metric space. A subset F of X is closed in X if $X \setminus F$ is open in X.

Exercise. Show the following:

(1) $\emptyset$ and X are closed in X.

(2) If U and V are closed in X, so is $U \cup V$.

(3) If $\{U_\alpha\}$ is a family of closed sets in X, so is $\cap U_\alpha$.

(4) Every subset of ω is closed in ω.

Definition. A subset A of a metric space X is called a G_δ set in X if we can write $A = \cap_n U_n$, U_n's open in X. A subset A of X is called an F_σ set in X if we can write $A = \cup_n F_n$, F_n's closed in X.

Exercise. Let (X, d) be a metric space and $F \subset X$ closed. For each $n \geq 1$, set

$$U_n = \{y \in X : \exists x \in A(d(x, y) < 1/n)\}.$$

Show that each U_n is open and $F = \cap U_n$. Conclude that every closed set in a metric space X is a G_δ set in X and every open set in X is an F_σ.

Definition. A subset D of X is called dense if $D \cap U \neq \emptyset$ for every non-empty open set U. A metric space X is called separable if it has a countable dense subset. A metric space that is complete and separable is also called a Polish space.

Exercise. Show that the metric spaces ω, ω^ω, $\{0,1\}^\omega$, $\mathbb{R}^n$ and $[0,1]^n$ defined in the beginning of this section are all separable.

Definition. Let (X, d) be a metric space and $\mathcal{B}$ a family of open subsets of X. We call $\mathcal{B}$ a base (for the toplogy of) X if

$$\forall \text{open } U \forall x \in U \exists B \in \mathcal{B}(x \in B \subset U).$$

Equivalently, $\mathcal{B}$ is a base for X if every set in $\mathcal{B}$ is open and if every open set X is the union of a subfamily of $\mathcal{B}$. We call X second countable if X has a countable base.

Exercise. For $s \in \omega^n$, set

$$\Sigma(s) = \{\alpha \in \omega^\omega : \alpha | n = s\}.$$

Show that $\{\Sigma(s) : s \in \omega^{<\omega}\}$ is a base for ω^ω.

Exercise. Show that a metric space is second countable if and only if it is separable.

Product metric. Let (X_n, d_n), $n \geq 1$, be a sequence of metric spaces and $X = \Pi_n X_n$. For $(x_n), (y_n) \in X$, define

$$d((x_n), (y_n)) = \sum_n \frac{d_n(x_n, y_n) \wedge 1}{2^n}.$$

Then d is a metric on X. Convergence in $(X.d)$ is the "co-ordinatewise convergence." Further, X is separable (complete) if and only if each X_n is so.

Exercise. Let X and Y be metric spaces and $f : X \to Y$ a continuous map. Show that its graph

$$\text{gr}(f) = \{(x, y) : X \times Y : y = f(x)\}$$

is closed in X.

Here is a result which makes ω^ω the single most important Polish space, at least as far as the Descriptive set Theory is concerned.

Theorem. Every Polish space is a continuous image of ω^ω.

Proof. Let (X, d) be a complete separable metric space. Fix a countable base $\mathcal{B}$ for X containing X. For any $A \subset X$, set

$$\text{diameter}(A) = \sup\{d(x, y) : x, y \in A\}.$$

Assume, without loss of generality, diameter$(X) < 1$. Let $\omega^{<\omega}$ denote the set of all finite sequences of natural numbers including the empty sequence e. Set

$$U_e = X.$$

Suppose for a finite sequence s we have defined an $U_s \in \mathcal{B}$ of diameter $< 2^{-|s|}$, where $|s|$ denotes the length of s.

Get a sequence $\{U_{sn}\}$ in $\mathcal{B}$ that covers U_s. Further assume that the diameter of each U_{sn} is $< 2^{-|s|-1}$ and the closure of each U_{sn} is contained in U_s.

Note that for any $\alpha \in \omega^\omega$, $\cap_n U_{\alpha|n}$ is a singleton, where $\alpha|n$ denotes the restriction of α to $n = \{0, 1, \cdots, n-1\}$. Set $f(\alpha)$ to be the unique point in $\cap_n U_{\alpha|n}$. Check that $f : \omega^\omega \to X$ is a continuous map from ω^ω onto X.

Later we shall state one more result so that we can develop the entire theory for ω^ω only.

Standard Borel Spaces

Definition. Let X be a non-empty set. An algebra $\mathcal{F}$ on X is a family of subsets of X such that

(1) $\emptyset$, X are in $\mathcal{F}$.

(2) If A is in $\mathcal{F}$, so is $X \setminus A$, i.e., $\mathcal{F}$ is closed under complementations.

(3) If $A, B \in \mathcal{F}$, so does $A \cup B$, i.e., $\mathcal{F}$ is closed under finite unions.

Definition. Let X be a non-empty set. A σ-algebra $\mathcal{A}$ on X is a family of subsets of X such that

(1) $\emptyset$, X are in $\mathcal{A}$.

(2) If A is in $\mathcal{A}$, so is $X \setminus A$.

(3) If $\{A_n\}$ is a sequence is in $\mathcal{A}$, $\cup_n A_n \in \mathcal{A}$, i.e., $\mathcal{A}$ is closed under finite unions.

Exercise. Show that every σ-algebra is closed under countable intersections.

It is easy to see that the intersection of a family of σ-algebras is a σ-algebra. So, given any family $\mathcal{G}$ of subsets of a set X, there is a smallest σ-algebra on X containing $\mathcal{G}$.

Definition. The smallest σ-algebra containing all open subsets of a metric space X is called the Borel σ-algebra of X. We shall write $\mathcal{B}_X$ to denote the Borel σ-algebra of X. Any set in $\mathcal{B}_X$ is called a Borel subset of X.

Here are a few very easy remarks.

(1) Since every subset of ω is open in X, every subset of of ω is Borel.

(2) Every closed set, every F_σ set and every G_δ set in a metric space is Borel in X.

(3) Let (X, d) be a metric space and $x \in X$. Since $\{x\}$ is closed, it is Borel in X. It follows that every countable subset of X is Borel.

Definition. Let X and Y be metric space. A map $f : X \to Y$ is called Borel measurable or simply Borel if for every open set U in Y, $f^{-1}(U)$ is Borel in X.

Exercise. Let X, Y and Z be metric spaces and $f : X \to Y$, $g : Y \to Z$ Borel maps. Show the following:

(1) For every Borel set B in Y, $f^{-1}(B)$ is Borel in X.

(2) The map $g \circ f : X \to Z$ is Borel.

(3) Assume, moreover, Y is second countable and $\mathcal{B}$ a countable base for Y. Then a map $h : X \to Y$ is

Borel if and only if $f^{-1}(B)$ is Borel in X for every $B \in \mathcal{B}$.

(4) If Y is second countable, show that $\mathrm{gr}(f)$ is Borel.

Lemma. Let X, Y be metric spaces and $f_n : X \to Y$ a Borel map, $n \geq 0$. Assume that for each $x \in X$, $\{f_n(x)\}$ is convergent and converges to, say $f(x)$. Then $f : X \to Y$ is Borel.

Proof. Let $U \subset Y$ be open. The for any $x \in X$,

$$f(x) \in U \Leftrightarrow \exists N \forall n \geq N (f_n(x) \in U).$$

Thus,

$$f^{-1}(U) = \cup_N \cap_{n \geq N} f_n^{-1}(U).$$

Since each f_n is Borel, for every open U in X, $f_n^{-1}(U)$ is Borel. Hence, $f^{-1}(U)$ is Borel.

Exercise. Let X, Y_n, $n \geq 1$, be metric spaces. Further assume that each Y_n is separable. Show that a map $f : X \to \Pi_n Y_n$ is Borel if and only if each $\pi_n \circ f : X \to Y_n$, $n \geq 1$, is Borel, where $\pi_n : Y \to Y_n$ is the projection function to Y_n.

Lemma. Let X be a metric space. Then $\mathcal{B}_X$ is the smallest class $\mathcal{C}$ of subsets of X that contains all open (or all closed sets) and that is closed under countable unions and countable intersections.

Proof. We shall consider the open case only because the closed case is proved similarly. The main reason for this result to be true is that every closed set in X is a G_δ in X. (For closed case, use that every open set is an F_σ.) Clearly $\mathcal{C} \subset \mathcal{B}_X$. So, the result will be proved if we show that $\mathcal{C}$ is closed under complementations. Set

$$\mathcal{D} = \{A \in \mathcal{C} : X \setminus A \in \mathcal{C}\}.$$

Since every closed set is a G_δ, every open set is in $\mathcal{D}$. Now take a sequence $\{A_n\}$ in $\mathcal{D}$. So for every n, both A_n and its complement $X \setminus A_n$ are in $\mathcal{C}$. We have

$$X \setminus \cup_n A_n = \cap_n (X \setminus A_n)$$

and

$$X \setminus \cap_n A_n = \cup_n (X \setminus A_n).$$

Since $\mathcal{C}$ is closed under countable unions and countable intersections, both $\cup_n A_n$ and $\cap_n A_n$ are in $\mathcal{D}$. Since $\mathcal{C}$ is the smallest family containing all open sets and closed under countable unions and countable intersections, it follows that $\mathcal{C} \subset \mathcal{D}$, i.e., $\mathcal{C}$ is closed under complementations.

As a corrolary to this lemma we get the following interesting result.

Theorem. Every standard Borel space is a continuous image of ω^ω.

Proof. Let (X, d) be a complete, separable metric space. Let $\mathcal{C}$ denote the family of all subsets of X that is continuous image of ω^ω. Since every closed subset of X is a complete separable metric space, they belong to $\mathcal{C}$. By the last lemma, it is now sufficient to show that for every sequence $\{A_n\}$ in $\mathcal{C}$, $\cup_n A_n, \cap_n A_n \in \mathcal{C}$. Towards showing it, fix a continuous onto map $f_n : \omega^\omega \to A_n$, $n \geq 0$.

Define $g : \omega^\omega \to X$ by

$$g(\alpha) = f_{\alpha(0)}(\alpha(1), \alpha(2), \ldots) \ \alpha \in \omega^\omega.$$

Check that g is a continuous map on ω^ω with range $\cup_n A_n$.

To show that $\cap_n A_n \in \mathcal{C}$, set

$$C = \{(\alpha_0, \alpha_1, \ldots) \in (\omega^\omega)^\omega : f_0(\alpha_0) = f_1(\alpha_1) = f_2(\alpha_2) = \ldots\}.$$

Check that C is a Polish space. Hence there is a continuous map h from ω^ω onto C. Now set

$$f(\alpha) = f_0(h(\alpha)(0)), \ \alpha \in \omega^\omega.$$

Then f is a continuous map from C onto $\cap_n A_n$.

Cardinalities of Standard Borel Spaces

Definition. A Borel subset of a Polish space is called standard Borel.

We prove the following important result in this section.

Theorem. Any uncountable metric space that is a continuous image of ω^ω is of the cardinality $\mathfrak{c}$. In particular, any uncountable standard Borel space is of the cardinality $\mathfrak{c}$.

Remark. This result was first proved by P. S. Alexandrov who, it seems, had thought that a counterexample to the Continuum Hypothesis can be found among standard Borel spaces. So, he was a bit disappointed to prove this result which is arguably the first important result in Descriptive Set Theory. In its proof the Souslin operation makes its first appearance. Incidentally, the Souslin operation was initially called Operation(A). But Alexandrov was not a favorite of Lusin and Souslin was. This seems to be the reason for Lusin changing the name of this important set-theoretic operation.

The proof needs the following result from topology.

Cantor-Bendixson Theorem. Let X be a second countable space. Then we can write $X = Y \cup Z$, where Y and Z are disjoint, Y is countable and open and Z has no isolated points.

Proof. Take a countable base $\mathcal{B}$ for X. Set

$$Y = \cup\{U \in \mathcal{B} : |U| \leq \aleph_0\},$$

and $Z = X \setminus Y$.

Proof of the Theorem. Let $f : \omega^\omega \to X$ be a continuous map with range uncountable. By the Axiom of Choice there is an uncountable subset Y of ω^ω such that $f|Y$ is one-to-one. By the Cantor-Bendixson theorem, without loss of generality, we can assume that Y has no isolated points. Let $2^{<\omega}$ be the set of all finite sequences of 0's and 1's including the empty sequence e. For each $s \in 2^{<\omega}$, we define closed set F_s in ω^ω satisfying the following properties:

(1) $F_s \cap Y \neq \emptyset$.
(2) Diameter of $F_s < 2^{-|s|}$.
(3) $F_{s\epsilon} \subset F_s$ for $\epsilon = 0$ or 1.
(4) $(s \neq t \wedge |s| = |t|) \Rightarrow F_s \cap F_t = \emptyset$.
(5) $(s \neq t \wedge |s| = |t|) \Rightarrow \overline{f(F_s)} \cap \overline{f(F_t)} = \emptyset$.

We assume the existence of such a system of closed sets and complete the proof first. For any $\alpha \in 2^\omega$, define $g(\alpha)$ to be the unique point in $\cap_n F_{\alpha|n}$. Now note that $f \circ g$ is a one-to-one function from 2^ω into the range of f. So, the cardinality of the range of f is $\mathfrak{c}$.

We define $\{F_s : s \in 2^{<\omega}\}$ by induction on the length of s. Set $U_e = F_e = \omega^\omega$. Note that U_e is open. Now let U_s be an open set whose closure is F_s and which intersects Y. Since Y has no isolated points, there exists at least two distint points in $Y \cap U_s$, say x_0 and x_1. Since f is one-to-one on Y, $f(x_0) \neq f(x_1)$. By continuity of f we get open sets $U_{s0} \ni x_0$ and $U_{s1} \ni x_1$ of diameters $< 2^{-|s|-1}$, with closures contained in U_s and such that $\overline{f(U_{s0})} \cap \overline{f(U_{s1})} = \emptyset$. Set $F_{s\epsilon} = \overline{U_{s\epsilon}}$, $\epsilon = 0$ or 1.

Remark. We have proved that any uncountable space that is a continuous image ω^ω contains a homeomorph of 2^ω. Hence, such spaces also contains a homeomorph of ω^ω.

Definition. Two metric spaces X and Y are called Borel isomorphic if there is a bijection $f : X \to Y$ such that both f and f^{-1} are Borel.

A very important result in theory of standard Borel spaces is the following:

Borel Isomorphism Theorem. Two standard Borel spaces are Borel isomorphic if and only if they are of the same the cardinalities. In particular, every uncountable Polish space (being of the cardinality $\mathfrak{c}$) is Borel isomorphic to ω^ω.

Probably, the simplest proof of it is given in

B. V. Rao and S. M. Srivastava, *An elementary proof of the Borel isomorphism theorem*, Real Analysis Exchange, 20(1), 1994-95, 1—3.

This proof is also presented in

S. M. Srivastava, **A Course on Borel Sets**, GTM 180, Springer.

Remark. We shall be interested in studying Borel sets and Projective sets in Polish spaces. Mainly because of the Borel isomorphism theorem, it is sufficient to develop the theory on ω^ω.

Hierarchy of Borel Sets.

The main goal of this section is to present a universal set argument. The idea is quite important.

Let X be a set and $\mathcal{F}$ a family of subsets of X. We put

$$\neg\mathcal{F} = \{A \subset X : X \setminus A \in \mathcal{F}\},$$

$$\mathcal{F}_\sigma = \{\cup_n A_n : A_n \in \mathcal{F}\}$$

and

$$\mathcal{F}_\delta = \{\cap_n A_n : A_n \in \mathcal{F}\}.$$

So, $\mathcal{F}_\sigma$ ($\mathcal{F}_\delta$) is the family of countable unions (resp. countable intersections) of sets in $\mathcal{F}$. The family of finite unions (finite intersections) of sets in $\mathcal{F}$ will be denoted by $\mathcal{F}_s$ (resp. $\mathcal{F}_d$). It is easily seen that

$$\mathcal{F} \subset \mathcal{F}_s \subset \mathcal{F}_\sigma,\ \mathcal{F} \subset \mathcal{F}_d \subset \mathcal{F}_\delta,$$

$$\mathcal{F}_\sigma = \neg(\neg\mathcal{F})_\delta,\ \text{and}\ \mathcal{F}_\delta = \neg(\neg\mathcal{F})_\sigma.$$

Let $X = \omega^\omega$ or any metric space. Let $\mathbf{\Sigma^0_1}$ denote the set of all open sets in ω^ω and $\mathbf{\Pi^0_1}$ denote the set of all closed sets. Note that $\mathbf{\Sigma^0_1}$ equals $\neg\mathbf{\Pi^0_1}$.

Finally we put

$\mathbf{\Delta^0_1} = \mathbf{\Sigma^0_1} \cap \mathbf{\Pi^0_1}$.

For ordinals $1 < \alpha < \omega_1$, we define $\mathbf{\Sigma^0_\alpha}$ to be the collection of those subsets A of X of the form $\cup_n(X \setminus B_n)$, where $B_n \in \mathbf{\Sigma^0_{\beta_n}}$ for some $\beta_n < \alpha$. Finally define $\mathbf{\Pi^0_\alpha}$ to be $\neg\mathbf{\Sigma^0_\alpha}$ and $\mathbf{\Delta^0_\alpha} = \mathbf{\Sigma^0_\alpha} \cap \mathbf{\Pi^0_\alpha}$.

Note that for $1 < \alpha < \omega_1$, $\mathbf{\Sigma^0_\alpha}$ equals $(\cup_{\beta<\alpha}\mathbf{\Pi^0_\beta})_\sigma$ and $\mathbf{\Pi^0_\alpha}$ equals $(\cup_{\beta<\alpha}\mathbf{\Sigma^0_\beta})_\delta$.

The next few results can be easily proved by induction on α.

Theorem. $\mathbf{\Sigma^0_\alpha}$ is closed under countable unions and finite intersections, $\mathbf{\Pi^0_\alpha}$ is closed under finite unions and countable intersections and $\mathbf{\Delta^0_\alpha}$ is closed under finite unions, finite intersections and complementations.

Theorem. Let X be any metric space.

(1) $\mathbf{\Sigma^0_\alpha} \subset \mathbf{\Pi^0_{\alpha+1}}$, $\mathbf{\Pi^0_\alpha} \subset \mathbf{\Sigma^0_{\alpha+1}}$.

(2) $\mathbf{\Sigma^0_\alpha} \cup \mathbf{\Pi^0_\alpha} \subset \mathbf{\Delta^0_{\alpha+1}}$.

Theorem. For any metric space X, $\mathcal{B}_X = \cup_{\alpha<\omega_1}\mathbf{\Sigma^0_\alpha}$ and also $\mathcal{B}_X = \cup_{\alpha<\omega_1}\mathbf{\Pi^0_\alpha}$

It is interesting to note that if X is an uncountable standard Borel space, the above hierarchy of Borel sets is strict, i.e., for every $1 \leq \alpha < \omega_1$, $\mathbf{\Sigma^0_\alpha} \neq \mathbf{\Pi^0_\alpha}$. The idea of its proof is also quite important. We shall prove the result for $X = \omega^\omega$ only. The result for general X follows quite easily from this.

So fix a countable base $\mathcal{B} = \{W_n\}$ for $X = \omega^\omega$. Define

$$U_1^\sigma = \{(\alpha, \beta) \in \omega^\omega \times \omega^\omega : \exists n(\beta \in W_{\alpha(n)})\}.$$

It is easily checked that U_1^σ is open in $\omega^\omega \times \omega^\omega$ and that for every open set V in ω^ω, there is an $\alpha \in \omega^\omega$ such that

$$V = \{\beta \in \omega : (\alpha, \beta) \in U_1^\sigma\}.$$

Such an universal open set produces a closed set that is not open:

Set

$$A = \{\alpha \in \omega^\omega : (\alpha, \alpha) \notin U^\sigma\}.$$

Note that A is closed in ω^ω. Suppose A is also open. Then get a $\gamma \in \omega^\omega$ such that

$$A = \{\beta \in \omega : (\gamma, \beta) \in U_1^\sigma\}.$$

Now, by the Cantor's diagonal argument, we see that

$$\beta \in A \Leftrightarrow \beta \notin A.$$

This contradiction proves that A is not open.

By setting

$$U_1^\delta = (\omega^\omega \times \omega^\omega) \setminus U_1^\sigma$$

we get an universal closed sets. Now by induction on α, we produce an universal set for each Borel pointclasses $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ and $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$. Then by the above diagonal argument, we shall arrive at our claim.

Suppose $1 < \alpha < \omega_1$ and that universal sets U_β^σ for $\mathbf{\Sigma}^0_{\boldsymbol{\beta}}$ and U_β^π for $\mathbf{\Pi}^0_{\boldsymbol{\beta}}$ have been defined for all $1 \le \beta < \alpha$.

We produce an universal set U_α^σ for $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ now. Fix a sequence of ordinals $\{\beta_n\}$, $1 \le \beta_n < \alpha$, such that $\alpha = \sup_n\{\beta_n + 1\}$. Fix a bijection $(k_0, k_1) \to \langle k_0, k_1 \rangle$ from $\omega \times \omega$ onto ω. Define $U_\alpha^\sigma \subset \omega^\omega \times \omega^\omega$ by

$$(\alpha, \beta) \in U_\alpha^\sigma \Leftrightarrow \exists k((\alpha(\langle k, 0\rangle), \alpha(\langle k, 1\rangle), \alpha(\langle k, 2\rangle), \ldots), \beta) \in U_{\beta_k}^\pi.$$

Check that U_α^σ is in $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ and that it is univeral for $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$.

Having defined U_α^σ, take $U_\alpha^\pi = (\omega^\omega \times \omega^\omega) \setminus U_\alpha^\sigma$.

Exercise. Let X be an uncountable standard Borel space. Show that for every $1 \le \alpha < \omega_1$, $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}} \neq \mathbf{\Pi}^0_{\boldsymbol{\alpha}}$.

Analytic and Coanalytic Sets.

Definition. Let X be a Polish space. A subset A of X is called analytic if there is a continuous map f from ω^ω onto A. A subset C of X is called coanalytic if $X \setminus C$ is analytic.

$\mathbf{\Sigma}^1_1$ will denote the pointclass of all analytic sets and $\mathbf{\Pi}^1_1$ for the pointclass of all conalytic sets. Further, we set

$\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$.

We have already seen that the family of all subsets of X that are continuous images of ω^ω is closed under countable unions and countable intersections and that it contains all Borel sets. Hence, we have the following theorem.

Theorem. $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are closed under countable unions and countable intersections. Further, $\mathbf{\Sigma}^1_1$ is closed under continuous images. Borel sets are simultaneously both analytic and coanalytic.

Theorem. Every uncountable analytic set is of cardinality $\mathfrak{c}$. Indeed, they contain a homeomorph of $\{0,1\}^\omega$.

A very curious question arises now: Is every analytic set Borel? Using the universal set argument, we show that this is not the case. We give some equivalent definitions of analytic sets first.

Proposition. Let X be a Polish space and $A \subset X$. The following statements are equivalent.

(1) A is analytic.

(2) A is the projection of a closed set in $X \times \omega^\omega$.

(3) A is the projection of a Borel set in $X \times \omega^\omega$.

(4) A is the projection of a Borel set in $X \times Y$ for some Polish space Y.

(5) A is the image of a standard Borel space under a Borel map.

Remark. Let $X = Y = Z = \omega^\omega$. There is a closed set C in $X \times (Y \times Z)$ that is universal for closed subsets of $Y \times Z$. Set

$$U = \pi_{X \times Y} C.$$

Then U is analytic and universal for all analytic subsets of Y. Also note that $W = (X \times Y) \setminus U$ is coanalytic and universal for coanalytic subsets of Y.

Example. Let U and W be as defined above. Set

$$A = \{\alpha \in \omega^\omega : (\alpha, \alpha) \in U\}.$$

Then A is analytic. If possible, suppose A is Borel. Then $B = \omega^\omega \setminus A$ is analytic. Note that

$$B = \{\alpha \in \omega^\omega : (\alpha, \alpha) \notin U\}.$$

There exists a $\gamma \in \omega^\omega$ such that

$$B = \{\beta \in \omega^\omega : (\gamma, \beta) \in U\}.$$

But then

$$\gamma \in B \Leftrightarrow \gamma \notin B.$$

Theorem. Let X be a Polish space, μ a continuous probability measure on X and $A \subset X$ analytic. Let $\epsilon > 0$. Then there is a compact $K \subset A$ such that $\mu(A \setminus K) < \epsilon$. In particular, analytic and coanalytic sets are universally measurable.

Proof. For $s = (n_0, \cdots, n_{k-1}) \in \omega^k$ and any $i \in \omega$, set

$$\Sigma^*(si) = \{\alpha \in \omega^\omega : \alpha|k = s \wedge \alpha(k) \leq i\}.$$

Let μ^* denote the outer measure induced by μ. Since $f(\Sigma^*(n)) \uparrow f(\Sigma)$, there exists n_0 such that

$$\mu^*(f(\Sigma) \setminus f(\Sigma^*(n_0))) < \epsilon 2^{-1}.$$

We get n_1 such that for each $i \leq n_0$,

$$\mu^*(f(\Sigma(i)) \setminus f(\Sigma^*(in_1))) < \epsilon 2^{-2}.$$

Proceeding inductively we get a sequence $n_0, n_1, \ldots$ such that forall k and for all $i_0 \leq n_0, \cdots, i_{k-1} \leq n_{k-1}$,

$$\mu^*(f(\Sigma(i_0, \cdots, i_{k-1})) \setminus f(\Sigma^*(i_0, \cdots, i_{k-1}, n_k))) < \epsilon 2^{-k-1}.$$

Let

$$L = \{\alpha \in \omega^\omega : \forall k(\alpha(k) \leq n_k)\}.$$

Then L is compact. Check that

$$\mu^*(A \setminus f(L)) < \epsilon.$$

Definition. A subset A of a Polish space X is said to have Baire Property if there is an open set U in X such that $A \Delta U$ is meager.

We shall omit the proof of the next couple of theorems.

Theorem. Sets with Baire property form a σ-algebra; it is the smallest σ-algebra containing all Borel sets and all meager sets.

Theorem. Every analytic and every coanalytic set has Baire property.

Separation and Reduction Principles.

By a pointclass $\mathbf{\Gamma}$ we shall mean a family of subsets of all Polish spaces. If $\mathbf{\Gamma}$ is a pointclass, we set

$\mathbf{\Delta} = \mathbf{\Gamma} \cap \neg\, \mathbf{\Gamma}$.

Definition. We say that $\mathbf{\Gamma}$ satisfies separation principle if for any two disjoint sets A and B in $\mathbf{\Gamma}$, there is a C in $\mathbf{\Delta}$ such that

$$A \subset C \wedge B \cap C = \emptyset.$$

Exercise. Show that if $\mathbf{\Gamma}$ is closed under countable unions and countable intersections and if it satisfies the separation principle, then for every sequence $\{A_n\}$ of pairwise disjoint sets in $\mathbf{\Gamma}$, there is a sequence $\{B_n\}$ of pairwise disjoint sets in $\mathbf{\Delta}$ such that $\forall n(A_n \subset B_n)$.

The next result is of fundamental importance in the theory of Borel sets.

First Separation Principle for Analytic Sets. (Souslin) Let X be a Polish space and A and B be disjoint analytic subsets. Then there is a Borel set C such that

$$A \subset C \wedge B \cap C = \emptyset.$$

Thus, $\mathbf{\Sigma_1^1}$ satisfies the separation principle.

Proof. We need a lemma first.

Lemma. Let $\{A_n\}$ and $\{B_m\}$ be a sequence of subsets of X such that there is no Borel set C satisfying

$$\cup_n A_n \subset C \wedge \cup_m B_m \cap C = \emptyset.$$

Then there is a n and a m such that there is no Borel set C_{nm} satisfying

$$A_n \subset C_{nm} \wedge B_m \cap C_{nm} = \emptyset.$$

Proof of the Lemma. Suppose for every n and m, a Borel set C_{nm} satisfying the above condition exist. Set

$$C = \cup_n \cap_{m \neq n} C_{nm}.$$

Then

$$A \subset C \wedge B \cap C = \emptyset.$$

Proof of the separation principle. Let $f : \omega^\omega \to A$ and $f : \omega^\omega \to A$ be continuous surjectios. Assume that there is no Borel set C such that

$$A \subset C \wedge B \cap C = \emptyset.$$

By applying the last lemma repeatedly and proceeding by induction, we get $\alpha, \beta \in \omega^\omega$ such that for every k there is no Borel set C satisfying

$$f(\Sigma(\alpha|k)) \subset C \wedge g(\Sigma(\beta|k)) \cap C = \emptyset.$$

Since A and B are disjoint, $f(\alpha) \neq g(\beta)$. Get disjoint open sets U and U in X such that $f(\alpha) \in U$ and $g(\beta) \in V$. By continuity of f and g, there is a k such that

$$f(\Sigma(\alpha|k)) \subset U \wedge g(\Sigma(\beta|k)) \subset V.$$

We have arrived at a contradiction.

Corollary. Every $\mathbf{\Delta^1_1}$ set is Borel.

Mokobodzki modified the above argument beautifully, and gave a similar proof of the following result.

Generalized Separation Principle. Let X be a Polish and $\{A_n\}$ a sequence of analytic subsets such that $\cap_n A_n = \emptyset$. Then there is a sequence $\{B_n\}$ of Borel sets such that $\forall n(B_n \supset A_n)$ and $\cap_n B_n = \emptyset$.

Theorem. Let X be a standard Borel space, Y a Polish space and $f : X \to Y$ a one-to-one Borel map. Then $f(X)$ is Borel.

Proof. Replacing X by the graph of f and f by the projection to Y, without any loss of generality, we assume that f is continuous. Since every standard Borel space is a one-to-one continuous image of a closed subset of ω^ω, without any loss of generality, we assume that X is a closed subset of ω^ω. We shall prove the result for $X = \omega^\omega$ only. The proof can be easily modified for closed subsets of ω^ω.

By the induction on the length of $s \in \omega^{<\omega}$, we define a system $\{B_s : s \in \omega^{<\omega}\}$ of Borel sets in Y such that for every $s, t \in \omega^{<\omega}$ and every $n \in \omega$, the following conditions are satisfied:

(1) $|s| \neq |t| \Rightarrow B_s \cap B_t = \emptyset$.

(2) $B_{sn} \subset B_s$.

(3) $f(\Sigma(s)) \subset B_s \subset \overline{f(\Sigma(s))}$.

Since f is one-to-one and continuous, $\{f(\Sigma(n))\}$ is a sequence of pairwise disjoint analytic sets in Y. Hence, by the separation principle, there is a sequence of pairwise disjoint Borel sets, $B_n \supset f(\Sigma(n))$. Replacing B_n by $B_n \cap \overline{f(\Sigma(n))}$, we see that the last condition is also satisfied by sequences of length 1.

Suppose B_s has been defined. Note that $\{B_s \cap f(\Sigma(sn))\}$ is a sequence of pairwise disjoint analytic sets in Y. Hence, by the separation principle, there is a sequence of pairwise disjoint Borel sets, $B_{sn} \supset f(\Sigma(sn))$. Replacing B_{sn} by $B_s \cap \overline{f(\Sigma(sn))}$, we see that the last condition is also satisfied by each sn.

Now observe that

$$f(\omega^\omega) = \cap_n \cup_{|s|=n} B_s.$$

Weak Reduction Principle for Coanalytic Sets. Let X be a Polish space, $\{C_n\}$ a sequence of coanalytic sets with $B = \cup_n C_n$ Borel. Then there exist pairwise disjoint Borel sets $B_n \subset C_n$ such that $\cup_n B_n = \cup_n C_n$.

Proof. Get Borel sets $D_n \supset B \setminus C_n$ such that $\cap_n D_n = \emptyset$. Now take $E_n = B \setminus D_n$, $n \geq 0$. Set $B_0 = E_0$ and for $n > 0$, $B_n = E_n \setminus \cup_{m<n} E_n$.

Exercise. Let X and Y be Polish spaces and $f : X \to Y$ any map. The following conditions are equivalent:

(1) f is Borel.

(2) The graph of f is Borel.

(3) The graph of f is analytic.

Projective Hierarchy.

We fix some notation first. Let X and Y be Polish spaces and $P \subset X \times Y$. We define subsets $Q = \exists^Y P$ and $R = \forall^Y P$ of X by

$$Q = \{x \in X : \exists y \in Y((x, y) \in P)\}$$

and

$$R = \{x \in X : \forall y \in Y((x, y) \in P)\}.$$

Note that

$$\exists^Y P = X \setminus \forall^Y((X \times Y) \setminus P)$$

and

$$\forall^Y P = X \setminus \exists^Y((X \times Y) \setminus P).$$

Let $\mathbf{\Gamma}$ be a pointclass, i.e., it is a family of subsets of all Polish spaces. For any Polish space Y, we define

$$\exists^X \mathbf{\Gamma} = \{\exists^Y P : P \in \mathbf{\Gamma}\ \}$$

and

$$\forall^X \mathbf{\Gamma} = \{\forall^Y P : P \in \mathbf{\Gamma}\ \}.$$

Note that

$$\mathbf{\Sigma^1_1} = \exists^{\omega^\omega} \mathbf{\Delta^1_1}$$

and

$$\mathbf{\Pi^1_1} = \forall^{\omega^\omega} \mathbf{\Delta^1_1}.$$

The projective hierarchy consists of the pointclasses $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$, $n \geq 1$, defined by induction as follows:

$$\mathbf{\Sigma}^1_{n+1} = \exists^{\omega^\omega} \mathbf{\Pi}^1_n$$

and

$$\mathbf{\Pi}^1_{n+1} = \forall^{\omega^\omega} \mathbf{\Sigma}^1_n.$$

Finally, we put

$$\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Delta}^1_n.$$

Projective classes have following closure properties.

(1) $\mathbf{\Sigma}^1_n = \neg \mathbf{\Pi}^1_n$, $\mathbf{\Pi}^1_n = \neg \mathbf{\Sigma}^1_n$.

(2) Let $\mathbf{\Gamma}$ be any of the projective class $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ or $\mathbf{\Delta}^1_n$. Then $\mathbf{\Gamma}$ is closed under countable unions and countable intersections. $\mathbf{\Delta}^1_n$ is closed under complementations. Furthr, assume that X and Y are Polish spaces, $f : X \to Y$ a Borel map and $A \subset Y$ is in $\mathbf{\Gamma}$. Then $f^{-1}(A) \in \mathbf{\Gamma}$.

(3) For every Polish space Y, $\mathbf{\Sigma}^1_n$ is closed under $\exists^Y$ and $\mathbf{\Pi}^1_n$ is closed under $\forall^Y$.

(4) $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subset \mathbf{\Delta}^1_{n+1}$.

Universal Projective Sets.

In this section we prove that every uncountable Polish space contains a set in $\mathbf{\Sigma}^1_n$ that is not in $\mathbf{\Pi}^1_n$. The proof goes by the usual universal set argument whose existence is based on the fact that a subset A of X is analytic if and only if it is the projection of a closed set in $X \times \omega^\omega$, i.e., $\mathbf{\Sigma}^1_1 = \exists^{\omega^\omega}\mathbf{\Pi}^0_1$ and that $\mathbf{\Pi}^0_1$ admits universal sets.

Theorem. Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$, $n \geq 1$. Then for every Polish space X, there is a set $U \subset \omega^\omega \times X$ in $\mathbf{\Gamma}$ which is universal for $\mathbf{\Gamma}$ subsets of X.

Proof. Let $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^1_1$. Let $W \subset \omega^\omega \times (X \times \omega^\omega)$ be a closed set that is universal for the family of closed subsets of $X \times \omega^\omega$. Then

$$U = \exists^{\omega^\omega} W = \{(\alpha, x) : \exists\beta((\alpha, x, \beta) \in U)\}$$

is in $\mathbf{\Sigma}^1_1$ and universal for $\mathbf{\Sigma}^1_1$ subsets of X.

If $U \subset \omega^\omega \times X$ is in $\mathbf{\Sigma}^1_n$ and universal for $\mathbf{\Sigma}^1_n$ subsets of X, then $(\omega^\omega \times X) \setminus U$ is in $\mathbf{\Pi}^1_n$ and universal for $\mathbf{\Pi}^1_n$ subsets of X.

If $U \subset \omega^\omega \times (X \times \omega^\omega)$ is in $\mathbf{\Pi}^1_n$ and universal for $\mathbf{\Pi}^1_n$ subsets of $X \times \omega^\omega$, then $\exists^{\omega^\omega} U$ is in $\mathbf{\Sigma}^1_{n+1}$ and universal for $\mathbf{\Sigma}^1_{n+1}$ subsets of X.

Use Cantor's diagonal argument in the following two exercises.

Exercise. Show that for every $n \geq 1$, there is a subset of ω^ω that is in $\mathbf{\Sigma}^1_n$ but not in $\mathbf{\Pi}^1_n$. Conclude this for all uncountable Polish spaces X.

Exercise. Let $n \geq 1$. Show that there is no $U \subset \omega^\omega \times \omega^\omega$ in $\mathbf{\Delta}^1_n$ that is universal for $\mathbf{\Delta}^1_n$ subsets of ω^ω.

Some More Concepts And Results.

Definition. Let $S \subset X$, X any set. An ordinal valued map φ on S is called a norm on S.

Let $\mathbf{\Gamma}$ be a pointclass.

Definition. Let X be Polish and $S \in \mathbf{\Gamma}$. A norm φ on S is called a $\mathbf{\Gamma}$-norm on S, if there exists binary relations $\leq_\varphi^{\mathbf{\Gamma}}$ in $\mathbf{\Gamma}$ and $\leq_\varphi^{\neg\mathbf{\Gamma}}$ in $\neg\mathbf{\Gamma}$ such that for every $y \in S$,

$$(x \in S \wedge \varphi(x) \leq \varphi(y)) \Leftrightarrow x \leq_\varphi^{\mathbf{\Gamma}} y \Leftrightarrow x \leq_\varphi^{\neg\mathbf{\Gamma}} y.$$

The following is a non-trivial result.

Theorem. Every $\mathbf{\Pi}_1^1$ set S in a Polish space admits a $\mathbf{\Pi}_1^1$-norm $\varphi : S \to \omega_1$.

Proposition. Let S be a $\mathbf{\Pi}_1^1$ set and $\varphi : S \to \omega_1$ a $\mathbf{\Pi}_1^1$ norm. Then for every $\alpha < \omega_1$, the set

$$S_\alpha = \{x \in S : \varphi(x) = \alpha\}$$

is Borel.

Note that in the above proposition, $S = \cup_{\alpha<\omega_1} S_\alpha$. Further, if S is $\mathbf{\Pi}_1^1$ and not Borel, then $\{\alpha < \omega_1 : S_\alpha \neq \emptyset\}$

is unbounded in ω_1. It is a little hard to show that if S is Borel then $S_\alpha \neq \emptyset$ only for countably many α's.

Theorem. Let S be a $\mathbf{\Pi^1_1}$ set. Then S is either countable or of cardinality $\aleph_1$ or of cardinality $\mathfrak{c}$.

Exercise. Let S be a $\mathbf{\Sigma^1_2}$ set. Show that S is either countable or of cardinality $\aleph_1$ or of cardinality $\mathfrak{c}$.

Definition. Let $S \subset X$, X a Polish space. A scale on S is a sequence $\{\varphi_n\}$ of norms on S such that whenever $x_i \in S$, $x_i \to x$, and for each n, $\{\varphi_i(x_n)\}$ is eventually a constant, say μ_n, $x \in S$ and $\forall n(\varphi_n(x) \leq \mu_n)$.

Definition. Let $S \in \mathbf{\Gamma}$. A scale $\{\varphi_n\}$ on S is called a $\mathbf{\Gamma}$-scale if each φ_n is a $\mathbf{\Gamma}$-norm on S.

The next result is also non-trivial. It was implicit in a proof given by Kondo. Moschovakis and his colleagues did quite a bit of work to make things quite clear.

Theorem. Every $\mathbf{\Pi^1_1}$ set admit a $\mathbf{\Pi^1_1}$-scale.

We now state an important consequence of the above theorem.

Definition. A pointclass $\mathbf{\Gamma}$ is said to have uniformization property if for every $S \subset X \times Y$, X, Y Polish, in $\mathbf{\Gamma}$, there is a $G \subset S$ in $\mathbf{\Gamma}$ such that

$$\forall x \in \pi_X(S) \exists ! y((x, y) \in G),$$

where $\pi_X : X \times Y \to X$ is the projection map and $\exists !$ means "there exists a unique."

Theorem (Kondo) $\mathbf{\Pi^1_1}$ has the uniformization property.

Exercise. Show that $\mathbf{\Sigma^1_2}$ has the uniformization property.

We have stated some of the important results that can be proved in $ZF + DC$. We require set-theoretic hypothesis to extend these results for higher projective classes. Professor Benedikt Lowe will speak on this in this school.