B.Math I. Probability II. Final Exam.
April. 30. 10:00 AM- 1:00 PM.

Answer any six of the eight problems. Each problem carries ten points. (A problem with two questions will carry five points each.)

You may use any result that is discussed in the class (except in problem 5), but you need to state and explain it.

1. The servicing of a machine requires two steps. The respective times needed for these two steps being independent and exponentially distributed with respective mean values 0.2 and 0.3. If the repair man has 30 machines to service, approximate the probability that all the work can be completed in 10 hours.

Determine the time t so that the probability that the repair person finishes the 30 jobs in the preceding part within time t is approximately equal to 0.95.

(Recall that the mean and variance of an exponential random variable with parameter λ are respectively $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$.)

2. Suppose that $\xi_1, \xi_2, \ldots$are independent and identically distributed random variables such that $E\left[\xi_1\right] = \mu$ and $E\left[\xi_1^2\right] < \infty$. Define

$$X_k = \rho X_{k-1} + \xi_k, \quad k \geq 1, \text{ and } \quad X_0 = 0$$

where $|\rho| < 1$. Let $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. Use a suitable version of the Central Limit Theorem to show that $\sqrt{n}\left(\overline{X}_n - \vartheta\right)$ converges in distribution to a normal distribution, where $\vartheta = \frac{\mu}{1-\rho}$. Obtain the variance of the limiting normal distribution.

3. Suppose that 3 white and 3 black balls are distributed in two urns in such a way that each contains 3 balls.

We say that the system is in state i if the first urn contains i white balls, $i = 0, 1, 2, 3$. At each stage one ball is drawn from each urn and the ball drawn from the first urn is placed in the second whereas the ball drawn from the second urn.is placed in the first. Let X_n denote the state of the system after the nth stage.

(a): Compute the transition probabilities of the Markov Chain $X_n, n \geq 0$.

(b): Obtain the stationary probabilities.

4. Show that in a Markov chain if the state i is recurrent and $i \to j$, then $i \leftrightarrow j$ and j is also recurrent.

5. Consider a Markov chain with the following transition probability matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

where $a_j = p(1-p)^{j-1}$.

(a): Show that stationary probabilities π_j, $j = 1, 2, ...$, satisfy the following equations:

$$\pi_1 = p\pi_1 + p\pi_2$$

and

$$\pi_j = (1-p)\pi_{j-1} + p\pi_{j+1}, \;\; j \geq 2.$$

(b): Deduce that

$$\pi_{j+1} = \frac{1-p}{p}\pi_j, \;\; j \geq 1$$

and hence determine π_j for $j \geq 1$.

6. Suppose that $X_t, t \geq 0$, is a (homogeneous) Markov process with state space $\{0, 1, 2, ...\}$ such that $p_{ij}(\Delta) = P(X_{t+\Delta} = j | X_t = i)$ satisfies

$$p_{ij}(\Delta) = \begin{cases} \lambda\Delta + o(\Delta) & \text{if } \; j = i+1 \\ 1 - \lambda\Delta + o(\Delta) & \text{if } \; j = i \end{cases}.$$

where $\lambda > 0$. Show directly that X_t is a Poisson process.

7. There are m automatic machines serviced by an operator. Suppose that if at time t a machine is in a working state, then the probability that it will break down and call for service in the time interval $(t, t+\Delta)$ is $\lambda\Delta + o(\Delta)$ with $\lambda > 0$. Assume that the number of machines that the operator is capable of servicing over the interval $[0, t]$ is a Poisson process with intensity parameter $\mu > 0$. Further assume that the machines work independently and that the operator is busy if there is a machine on the waiting line. Denote by X_t the number of machines working at the moment t.

(a): Show that

$$P(X_{t+\Delta} = j | X_t = i) = \begin{cases} \mu\Delta + o(\Delta) & \text{if } \; j = i+1 \\ i\lambda\Delta + o(\Delta) & \text{if } \; j = i-1 \\ 1 - \mu\Delta - i\lambda\Delta + o(\Delta) & \text{if } \; j = i \end{cases}, \text{ for } \; i = 0, ..., m-1,$$

and

$$P(X_{t+\Delta} = j | X_t = m) = \begin{cases} m\lambda\Delta + o(\Delta) & \text{if } \; j = m-1 \\ 1 - m\lambda\Delta + o(\Delta) & \text{if } \; j = m \end{cases}.$$

(b): Show that

$$\lim_{t\to\infty} p_{ij}(t) = \frac{(\mu/\lambda)^j \,/\, j!}{\sum_{k=0}^{m} \frac{(\mu/\lambda)k}{k!}}, \;\; j = 0, 1, ..., m.$$

8. At each moment $t \geq 0$, a particle can be in any one of the states $0, 1, ..., m$. If it is in either one of the states 0 or m, it will remain there forever, that is, 0 and m are absorbing states. Assume that

$$P\left(X_{t+\Delta} = j | X_t = i\right) = \begin{cases} \lambda\Delta + o\left(\Delta\right) & \text{if } j = i + 1 \\ \mu\Delta + o\left(\Delta\right) & \text{if } j = i - 1 \\ 1 - \mu\Delta - \lambda\Delta + o\left(\Delta\right) & \text{if } j = i \end{cases}, \quad i = 1, ..., m-1,$$

where $\lambda > 0$, $\mu > 0$. Find the absorbing probabilities $\lim_{t\to\infty} p_{i0}\left(t\right)$ for $i = 1, ..., m - 1$. Discuss the situation under $m \to \infty$.