Algebra 3 Final Examination

Write your answers precisely and justify everything completely. You may use any results proved in the textbook (by making clear which result you are using) but no others. You must submit your cheat sheet at the end of the exam. All rings are commutative with a multiplicative identity 1.

0. **[4 points]** a) For ideals I and J in a ring R, suppose that the natural ring homomorphism from R to $R/I \times R/J$ is an isomorphism. Show that $I + J = R$ and that the intersection of I and J is 0.

[Addendum for 1 point] Note that the natural map in problem 0 is also a homomorphism of R-modules. (why?) Now suppose you are given only that there *exists* a bijection between R and $R/I \times R/J$ which is both a ring homomorphism as well as an R-module homomorphism. Why do the conclusions in part a still remain valid?

1. **[8 points]** Let R be a nonzero ring and I an ideal of R.
a) If I is a free R-module, show that $I = zR$ where z is a nonzero divisor of R.
b) If R/I is a free R module, show that $I = 0$.
c) If R/I is a free R module for every ideal I of R, what can you say about R?

2. **[8 points]** a) Let A be the 3 by 3 matrix with rows (6, 2, 15) (0, 4, 10) and (0, 0, 8). Consider the abelian group G presented by A. (i.e., G is the cokernel of the map between two free abelian groups given by A.) Write G as the direct sum of k nonzero abelian groups where k is as large as possible.
b) For which integers n can G be made into a module over Z/nZ?

3. **[8 points]** How many similarity classes of complex matrices are there whose characteristic polynomial is $(x - 2)^6$? Show that there is no polynomial p(x) such that there are precisely 13 similarity classes of complex matrices whose characteristic polynomial is p(x) .

4. **[8 points]** Show that a square matrix with entries in a field F is diagonalizable if and only if its minimal polynomial factors into distinct linear factors in F[x]. (If you have trouble, you can modify the problem and take F to be the field of complex numbers.)

There are 2 more problems in the test, worth 12 points each. If you wish you may attempt them before the lunch or after. But if you continue now, you may not leave till people who have gone for lunch come back!

5. [12 points] For each of the following, either show that the given element is irreducible in the given ring or factor it into irreducibles.

a) 30 in the ring of Gaussian integers Z[i]

b) $x^{50} - x^{49} - 9x^2 + 15x - 6$ in Z[x]

c) $1234567x^4 - 5123476x^3 - 1543276x^2 + 7654321x - 1524367$ in Q[x]

d) $64x^6 + 32x^5 + 16x^4 + 8x^3 + 4x^2 + 2x + 1$ in Z[x]

6. [12 points] For each ring R given below, determine when the number of maximal ideals in R is finite. When this number is finite, determine it in terms of the relevant polynomial(s). Do the same about the number of prime ideals.

If you wish, you may use without proof results about the ring in question that were proved in the homework. But be sure to state clearly any such result.

a) R = F[x]/(p(x)), where F is a field and p(x) a polynomial in F[x].

In parts b and c, let C be the field of complex numbers and let f(x,y) and g(x,y) be polynomials in C[x,y]. You may find it easier to think in terms of the language of algebraic geometry.

b) R = C[x,y]/(f).

c) R = C[x,y]/(f, g)

Algebra final exam - Optional Part 2

A. Look again at problem 0. Suppose you are given only that there is a ring isomorphism between R and $R/I \times R/J$. Do the conclusions stay valid? What if you are instead given only that there is an R-module isomorphism between R and $R/I \times R/J$?

B. Find necessary and sufficient conditions on integers a, b, m and n for there to be an isomorphism between abelian groups $Z/mZ \times Z/nZ$ and $Z/aZ \times Z/bZ$. Your condition should be purely in terms of the integers themselves and possibly some gcd's/lcm's. Can you find such a condition if the number of factors in one of the products exceeds two?

C. True or false? If there is an abelian group isomorphism between the product of some Z/n_iZ and a product of some Z/m_jZ, then there is also a ring isomorphism between them.

D. Using the Jordan form, we know that any linear operator A on a finite dimensional complex vector space can be written as D + N, where D is diagonalizable, N is nilpotent and DN = ND. Recall that uniqueness of D and N was an easy consequence of the following two results.

a) D and N can be expressed as polynomials in A.

b) Two commuting diagonalizable operators on a finite dimensional vector space can be simultaneously diagonalized in a suitable basis.

Can you prove these?

I am still waiting for you to provide me a complete classification of rings of cardinality 8, because I know that you can do it. When (not if) you finish it as a class, you should show it to me.