INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
M.MATH - Second Year, Second Semester, 2004-05
Statistics, Midterm Examination, March 14, 2005
Marks: 50; Time: 2 hours

1. (10) Let Z_1, Z_2 be i.i.d. $N(0,1)$ and $0 < \rho < 1$. Define $X_1 = Z_1$, $X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$. Find X_3 such that

$$\text{Cov}\ (X_1, X_2, X_3) = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.$$

2. (15) Let $\mu_i \in R^1$, $1 \le i \le p$, $-1 < \rho < 1$, and $\sigma^2 > 0$ be unknown parameters, and $\epsilon_i \sim N(0,1)$, $i = 1, \ldots, p$ be independent. Let $\mu = (\mu_1, \ldots, \mu_p)'$, $\epsilon = (\epsilon_1, \ldots, \epsilon_p)'$, and $\Sigma = \sigma^2 [(1-\rho) I_p + \rho 1 1']$. Let $X = \Sigma^{1/2} \epsilon + \mu$.
(a) Find the probability distribution of X.
(b) Find the determinant and inverse of $(1-\rho) I_p + \rho 1 1'$.
(c) Consider a random sample of size n from the distribution of X, $n > 1$. Find the maximum likelihood estimates of all the unknown parameters.

3. (15) Let $A \sim W_p(k, \Sigma)$ and let A_i and Σ_i denote the block sub-matrices consisting of the first i rows and columns of A and Σ, respectively. Define

$$v_1 = \frac{a_{11}}{\sigma_{11}}, \qquad v_i = \frac{|A_i||\Sigma_{i-1}|}{|A_{i-1}||\Sigma_i|}, \quad i = 2, \ldots, p.$$

(a) Show that $v_1, \ldots, v_p$ are independent χ^2 random variables and v_i has $k-i+1$ degrees of freedom.
(b) Show that $|A|/|\Sigma|$ is the product of p independent χ^2 random variables with the ith having degrees of freedom $k - i + 1$.

4. (10) Let X_1, X_2, X_3 be i.i.d. $N_p(\mu, \Sigma)$ and define $Y_1 = X_1 + X_2$, $Y_2 = X_2 + X_3$ and $Y_3 = X_1 + X_3$.
(a) Find the conditional distribution of Y_1 given Y_2;
(b) Find the conditional distribution of Y_1 given Y_2 and Y_3.