INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
M.MATH - Second Year, Second Semester, 2004-05
Statistics, Final Examination, May 17, 2005

1. (12) Let $\mu_i \in R^1$, $1 \leq i \leq p$, $0 < \rho < 1$, and $\sigma^2 > 0$ be unknown parameters, and $\epsilon_i \sim N(0, 1)$, $i = 1, \ldots, p$ be independent. Let $\mu = (\mu_1, \ldots, \mu_p)'$, $\epsilon = (\epsilon_1, \ldots, \epsilon_p)'$, and $\Sigma = \sigma^2 [(1-\rho)I_p + \rho 11']$. Let $X = \Sigma^{1/2}\epsilon + \mu$.
(a) Find the probability distribution of X.
(b) Find the population principal components and their variances.

2. (8) Consider a random sample of size $n > 2$ from $N_2(\mu, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Find the Generalized Likelihood Ratio Test for $H_0 : \rho = 0$.

3. (10) Let X be a random p-vector and Y be a random q-vector with the covariance matrix

$$Cov \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix},$$

where $\Sigma_{11} = (1-\alpha)I_p + \alpha 11'$, $\Sigma_{12} = \beta 11'$, and $\Sigma_{22} = (1-\gamma)I_q + \gamma 11'$. Find the canonical correlations between X and Y and also the canonical variables.

4. (8) Consider two bivariate normal populations $N_2(\mu, \Sigma_1)$ and $N_2(\mu, \Sigma_2)$, where $\Sigma_1 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ and $\Sigma_2 = \begin{pmatrix} \tau_1^2 & 0 \\ 0 & \tau_2^2 \end{pmatrix}$. If $\sigma_1^2 > \tau_1^2$ and $\sigma_2^2 > \tau_2^2$, show that the boundary of the maximum likelihood discriminant rule is an ellipse.

5. (7) Consider the multivariate one-way classification

$$X_{ij} = \mu_i + \epsilon_{ij}, \qquad \epsilon_{ij} \sim N_p(0, \Sigma), \text{ i.i.d.}, \qquad j = 1, \ldots n_i; \quad i = 1, \ldots, k.$$

Derive the Generalized Likelihood Ratio Test for $H_0 : \mu_i$ are all equal.

6. (5) If ρ_{ij} denotes the correlation coefficient between X_i and X_j, show that the partial correlation coefficient between X_1 and X_2 given X_3 is

$$\rho_{12.3} = \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}.$$