Indian Statistical Institute
Second Semester Examination 2003-2004
M.Math I Year & M.Stat II Year
Lie Groups, Lie Algebras, & Representation Theory

Time: 3 hrs Date: 10-05-2004 Max. Marks : 60

All questions carry equal marks. You may use your class-room notes and the text book by Bagchi-Madan-Sitaram-Tewari in the exam.

1. G is a compact group, μ is the Haar measure on G, and f a continuous function. $\exists\ x_0 \in G$ such that $f(x_0) \neq 0$. Prove that $0 < \int_G |f| d\mu < \infty$.

2. Let G be an *abelian* group. If $(\pi\ ,\ V)$ is an irreducible representation of G on a complex finite dimensional vector space V, prove that $\dim V = 1$

3. Let $U(n) = \{A;\ \ A$ an $n \times n$ complex matrix with $AA^* = I\}$ What is the Lie algebra of $U(n)$? Justify your answer.

4. If $u(n)$ denotes the Lie algebra of $U(n)$, prove that $\exp : u(n) \to U(n)$ is onto.

5. G is a compact linear Lie group. If $x, y \in G, x \neq y$, prove that there exists a complex finite dimensional irreducible representation π of G such that $\pi(x) \neq \pi(y)$.

6. Describe all the complex finite dimensional irreducible representations of $SU(2)$, upto equivalence. (You need not give a proof.)

7. f is an L^2-function on $SU(2)$. You are told that $\int_G f\pi_{ij}^{(n)} = 0$, $\forall\ i, j$ and $\forall\ n \geq 3$, and $\int_G f\pi_{ij}^{(n)} = 1$, $\forall\ n < 3$, and $\forall\ i, j$. What is the L^2-norm of f, i.e $||f||_2$? Justify your answer.

8. Let G be a connected linear Lie group and $\underline{g}$ its Lie algebra. π is a finite dimensional representation of G. If $\dot{\pi}(X) = 0$, $\forall X \in \underline{g}$, what can you say about π ? Justify your answer.

9. Let $G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix};\quad x, y, z \in I\!R \right\}$.

   What is the Lie algebra of G ? Justify your answer.