Your Signature _____

Instructions:
Please write your name on every page.
Maximum time is 3 hours. Please stop writing when you are asked to do so.

Penalties: -10 points when any instruction is not followed.

Score

1. (15)	
2. (15)	
3. (20)	
Total. (50)	

1. If X is a Binomial (n,p) random variable, $\lambda = np$, then, with $\phi(x) = (1+x)\log(1+x) - x$, $x \geq -1$, (and $\phi(x) = \infty$ for $x < -1$)

$$P(X \geq E(X) + t) \leq \exp(-\lambda\phi(\frac{t}{\lambda})) \leq \exp(-\frac{t^2}{2(\lambda + \frac{t}{3})}),\ t \geq 0$$

$$P(X \leq E(X) - t) \leq \exp(-\lambda\phi(\frac{t}{\lambda})) \leq \exp(-\frac{t^2}{2\lambda}),\ t \geq 0$$

2. Let Γ be any set. Suppose, for any $0 < M \leq |\Gamma|$, we chose $\Gamma_M \subset \Gamma$ uniformly from all the subsets of Γ of size M. Let $0 < p < 1$, Γ_p be as above and Q be any arbitrary property. Show that $P(\Gamma_M \in Q) = P(\Gamma_p \in Q | |\Gamma_p| = m)$.

3. Let ξ_i^k $i, k \geq 1$ be i.i.d. non-negative integer valued random variables with $P(\xi_i^k = m) = p_m$ for $m \geq 0$. Let $\mu = \sum_{m=0}^{\infty} m p_m > 1$. Let $\phi : [0,1] \to R$ be given by $\phi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k$.

(a) Show that ϕ is increasing, convex and that $\phi'(1) = \mu$

(b) Show that there is a unique $\rho < 1$ such that $\phi(\rho) = \rho$.

(c) Suppose $Z_0 = 1$ and for all $k \geq 1$, let $Z_k = \sum_{i=1}^{Z_{k-1}} \xi_i^k$ if $Z_{k-1} > 0$ and 0 otherwise. If $\theta_k = P(Z_k = 0)$ then show that $\theta_{k+1} = \phi(\theta_k)$.

(d) Assume (c). Show that $\lim_{k\to\infty} \theta_k = \rho$ and conclude that $P(Z_k = 0 \text{ for some } k) = \rho < 1$