Indian Statistical Institute, Bangalore

M.Math II Year, First Semester

Mid-Sem Examination

Basic Probability Theory

Time: 3 hours Date:01 Oct 2010 Instructor: B.Rajeev

Maximum Marks 90

1. Let $\{S_n\}$ be a simple random walk with probability $p, 0 < p < 1$ of moving to the right. Let $u_0 = 1$ and $u_n = P(S_n = 0), n \geq 1$.

   a) Compute u_n in terms of p and n. [4]

   b) Find a closed form expression for the generating function $U(s)$ of $\{u_n\}$. [6]

2. Suppose X is a non negative integer valued $r \cdot v \cdot$ with $P(X = k) =: p_k \quad k \geq 0$. Suppose T is integer valued and independent of X, with $P(T \geq n) = s^n, 0 < s < 1$. Compute $P(T \geq X)$ in terms of $\{p_k\}$ and s. [10]

3. Let $\{Z_n\}$ be a branching process with $Z_0 \equiv 1$ and off spring distribution given by $P(s) = q + ps, 0 < p, q < 1, p + q = 1$.

   a) Compute $P_n(s)$, the generating function of Z_n. [5]

   b) Let $T := \inf\{k \geq 0 : Z_k = 0\}$. Find $P(T = n)$. [7]

   c) If $Z_0 \equiv i$ where $i > 1$, find $P(T = n)$. [8]

4. Suppose X_1 and X_2 are independent real random variables.

   a) If $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a bounded measurable function, show that

   $$E\ g(X_1, X_2) = \int E\ g(x_1, X_2) P_{X_1}(dx_1)$$

   where P_{X_1} is the law of X_1. [5]

   b) Now assume that $X_i \sim N(0,1)\ i = 1, 2$. Show that X_1 has the same distribution as $-X_1$. [5]

   c) Show that $(X_1, X_1 + X_2)$ has the same distribution as $(X_1, X_1 - X_2)$. [10]

5. a) If X and Y are independent Poisson random variable with parameters λ and μ respectively, find the distribution of $X + Y$. [5]

   b) Let X have the Geometric distribution with parameter $p, 0 < p < 1$. Find the mean and variance of X. [5]

   c) Show that if $0 < q < p$ and if $E|X|^p < \infty$ then $E|X|^q < \infty$. [5]

6. a) Let X be a real random variable on $(\Omega, \mathcal{A}, \mathcal{P})$ and let $A_i \in \mathcal{A}$ be disjoint measurable sets with $\Omega = \bigcup_{1=1}^{\infty} A_i$. Suppose X is integrable show that

$$EX = \sum_{1=1}^{\infty} E(XI_{A_i}).$$

[5]

b) Suppose $\{X_i\}$ are real valued, $i \cdot i \cdot d$ random variables and α is a non negative integer valued $r \cdot v \cdot$ independent of the sequence $\{X_i\}$. Prove the Wald identity

$$E(\sum_{1=1}^{\alpha} X_i) = EX_1 \cdot E\alpha.$$

[10]

You may assume $E|X_1| < \infty$.

—end—