Inclusion systems and amalgamated product of product systems

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August 15, 2010

Abstract

Here we generalize the concept of spatial product, introduced by Skeide, of two product systems via a pair of normalized units. This new notion is called amalgamated tensor product of product systems of Hilbert spaces, and now the amalgamation can be done using a contractive morphism.

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Here we generalize the concept of spatial product, introduced by Skeide, of two product systems via a pair of normalized units. This new notion is called amalgamated tensor product of product systems of Hilbert spaces, and now the amalgamation can be done using a contractive morphism.

We parameterize all contractive morphism from a Type I product system to another Type I product system and compute index of amalgamated product through contractive morphisms.

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In the context of product system of Hilbert modules, Skeide [*The index of (white) noises and their product systems*. I.D.A.Q.P.] introduced spatial product of product systems as there is no natural tensor product operation on product system of Hilbert modules where it identifies the two reference units and index is additive under spatial product.

At the 2002 AMS summer conference on 'Advances in Quantum Dynamics' held at Mount Holyoke, R.T. Powers posed the following problem : Let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ be algebras of all bounded operators on two Hilbert spaces H and $\mathcal{K}$. Suppose $\phi = \{\phi_t : t \geq 0\}$ and $\psi = \{\psi_t : t \geq 0\}$ are two E_0 semigroups on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively and $U = \{U_t : t \geq 0\}$ and $V = \{V_t : t \geq 0\}$ are two strongly continuous semigroups of isometries which intertwine ϕ_t and ψ_t respectively. Consider the CP semigroup τ_t on $\mathcal{B}(H \oplus K)$ defined by

$$\tau_t \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \phi_t(X) & U_t Y V_t^* \\ V_t Z U_t^* & \psi_t(W) \end{pmatrix}.$$

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Later Powers [*Addition of spatial E_0-semigroups*, Operator algebras, quantization, and non commutative geometry, 281–298, Contemp. Math.] showed that it is not the tensor product.

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More generally, suppose $\alpha = (\alpha_t)_{t\geq 0}$, $\beta = (\beta_t)_{t\geq 0}$ are CP semigroups on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively. Also suppose
$\eta = (\eta_t)_{t\geq 0} : \mathcal{B}(K, H) \to \mathcal{B}(K, H)$ is a semigroup of bounded operators.
Then $\tau = (\tau_t)_{t\geq 0} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} := \begin{pmatrix} \alpha_t(X) & \eta_t(Y) \\ \eta_t(Z^*)^* & \beta_t(W) \end{pmatrix}$ is a CP semigroup on $\mathcal{B}(H \oplus K)$. Then what is the product system of τ is related to those of α and β ?

Inclusion system

Definition

An inclusion system (E, β) *is a family of Hilbert spaces* $E = \{E_t, t \in (0, \infty)\}$ *together with isometries* $\beta_{s,t}: E_{s+t} \to E_s \otimes E_t$, *for* $s, t \in (0, \infty)$, *such that* $\forall$ $r, s, t \in (0, \infty)$, $\quad (\beta_{r,s} \otimes 1_{E_t})\beta_{r+s,t} = (1_{E_r} \otimes \beta_{s,t})\beta_{r,s+t}$. *It is said to be a product system if further every* $\beta_{s,t}$ *is a unitary.*

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We will assume all inclusion systems are algebraic i.e. without any measurabilty structure

Inclusion system gives rise to product system

For $t \in \mathbb{R}_+$, let $J_t = \{(t_n, t_{n-1}, \ldots, t_1) : t_i > 0, \sum_{i=1}^{n} t_i = t, n \geq 1\}$.

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On J_t define a partial order $\mathbf{t} \geq \mathbf{s} = (s_m, s_{m-1}, \ldots, s_1)$ if for each i, $(1 \leq i \leq m)$ there exists (unique) $\mathbf{s}_i \in J_{s_i}$ such that $\mathbf{t} = \mathbf{s}_m \smile \mathbf{s}_{m-1} \smile \cdots \smile \mathbf{s}_1$.

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For $\mathbf{t} = (t_n, t_{n-1}, \ldots t_1)$ in J_t define $E_{\mathbf{t}} = E_{t_n} \otimes E_{t_{n-1}} \otimes \cdots \otimes E_{t_1}$.

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Theorem

Suppose (E, β) is an inclusion system. Let $\mathcal{E}_t = \text{indlim}_{J_t} E_{\mathbf{s}}$ be the inductive limit of $E_{\mathbf{s}}$ over J_t for $t > 0$. Then $\mathcal{E} = \{\mathcal{E}_t : t > 0\}$ has the structure of a product system of Hilbert spaces.

Units and morphisms

Units and morphisms

Definition

Let (E, β) *and* (F, γ) *be two inclusion systems. Let* $A = \{A_t : t > 0\}$ *be a family of linear maps* $A_t : E_t \rightarrow F_t$*, satisfying* $\|A_t\| \leq e^{tk}$ *for some* $k \in \mathbb{R}$*. Then* A *is said to be a* morphism *or a* weak morphism *from* (E, β) *to* (F, γ) *if*

$$A_{s+t} = \gamma^*_{s,t}(A_s \otimes A_t)\beta_{s,t} \quad \forall s, t > 0.$$

It is said to be a strong morphism *if*

$$\gamma_{s,t}A_{s,t} = (A_s \otimes A_t)\beta_{s,t} \quad \forall s, t > 0.$$

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Let (E, β) *be an inclusion system. Let* $u = \{u_t : t > 0\}$ *be a family of vectors with* $u_t \in E_t$*, for all* $t > 0$*, such that* $\|u_t\| \leq e^{tk}$ *for some* $k \in \mathbb{R}$ *and* $u \neq 0$*. Then* u *is said to be a* unit *or a* weak unit *if*

$$u_{s+t} = \beta_{s,t}^*(u_s \otimes u_t) \quad \forall s, t > 0.$$

It is said to be a strong unit *if*

$$\beta_{s,t}u_{s+t} = u_s \otimes u_t \quad \forall s, t > 0.$$

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Theorem

Let (E, β) *be an inclusion system and let* $(\mathcal{E}, B)$ *be the product system generated by it. Then the canonical map* $i_t : E_t \to \mathcal{E}_t$, $t > 0$ *is an isometric strong morphism of inclusion systems. Further* i^* *is an isomorphism between units of* $(\mathcal{E}, B)$ *and units of* (E, β).

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Theorem

Let $(E, \beta), (F, \gamma)$ be two inclusion systems generating two product systems $(\mathcal{E}, B), (\mathcal{F}, C)$ respectively. Let i, j be the respective inclusion maps. Suppose $A : (E, \beta) \to (F, \gamma)$ is a weak morphism then there exists a unique morphism $\hat{A} : (\mathcal{E}, B) \to (\mathcal{F}, C)$ such that $A_s = j_s^* \hat{A}_s i_s$ for all s. This is a one to one correspondence of weak morphisms. Further more, $\hat{A}$ is isometric/unitary if A is isometric/unitary.

With basic theory of inclusion systems and their morphisms in place, we look at inclusion systems arising from quantum dynamical semigroups. Let H be a a Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded operators on H. Let $\tau = \{\tau_t : t \geq 0\}$ be a quantum dynamical semigroup on $\mathcal{B}(H)$. For $t \geq 0$, let (π_t, V_t, K_t) be the minimal Stinespring dilation of τ_t. Now fix a unit vector $a \in H$, take

$$E_t = \overline{\text{span}}\{\pi_t(|a\rangle\langle g|)h : g, h \in H\} \subseteq K_t,$$

. Further fix an ortho-normal basis $\{e_k\}$ of H and define $\beta_{s,t} : E_{s+t} \to E_s \otimes E_t$ by

$$\beta_{s,t}(\pi_{s+t}(|a\rangle\langle g|V_{s+t}h) = \sum_k \pi_s(|a\rangle\langle g)V_s e_k \otimes \pi_t(|a\rangle\langle e_k|)V_t h.$$

Then (E, β) is an inclusion system.

Amalgamation

Suppose H and K are two Hilbert spaces and $D : K \to H$ is a linear contraction. Define a semi inner product on $H \oplus K$ by

$$\begin{aligned} \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_D &= \langle u_1, u_2 \rangle + \langle u_1, Dv_2 \rangle + \langle Dv_1, u_2 \rangle + \langle v_1, v_2 \rangle \\ &= \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tilde{D} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle, \end{aligned}$$

where $\tilde{D} := \begin{bmatrix} I & D \\ D^* & I \end{bmatrix}$. Note that as D is contractive, $\tilde{D}$ is positive definite. Take

$$N = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_D = 0 \right\}.$$

Then N is the kernel of bounded operator $\tilde{D}$ and hence it is a closed subspace of $H \oplus K$. Set G as completion of $(H \oplus K)/N$) with respect to norm of $\langle ., . \rangle_D$. We denote G by $H \oplus_D K$.

Now we consider amalgamation at the level of inclusion systems. Let (E,β) and (F,γ) be two inclusion systems. Let $D=\{D_s : s>0\}$ be a weak contractive morphism from F to E. Define $G_s := E_s \oplus_{D_s} F_s$ and $\delta_{s,t} := i_{s,t}(\beta_{s,t} \oplus_D \gamma_{s,t})$ where $i_{s,t} : (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t) \to G_s \otimes G_t$ is the map defined by

$$i_{s,t}\begin{bmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

and $(\beta_{s,t} \oplus_D \gamma_{s,t}) : E_{s+t} \oplus_{D_{s+t}} F_{s+t} \to E_s \otimes E_t \oplus_{D_s \otimes D_t} F_s \otimes F_t$ is the map defined by

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Proposition

Let $(G,\delta)=\{G_s,\delta_{s,t} : s,t>0\}$ be defined as above. Then $\{G,\delta\}$ forms an inclusion system

Universal properties of amalgamation

Definition

If $(\mathcal{E}, B), (\mathcal{F}, C)$, *and* $(\mathcal{G}, L)$ *are product systems generated respectively by* $(E, \beta), (F, \gamma)$, *and* (G, δ), *then* $(\mathcal{G}, L)$ *is said to be the amalgamated product of* $(\mathcal{E}, B)$ *and* $(\mathcal{F}, C)$ *via* D *and is denoted by* $\mathcal{G} =: \mathcal{E} \otimes_D \mathcal{F}$.

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Theorem

Suppose $(\mathcal{E}, W^{\mathcal{E}})$ and $(\mathcal{F}, W^{\mathcal{F}})$ are two product systems and let $C : (\mathcal{F}, W^{\mathcal{F}}) \to (\mathcal{E}, W^{\mathcal{E}})$ be a contractive morphism. Suppose $(\mathcal{G}, W^{\mathcal{G}})$ is the amalgamated product of $(\mathcal{E}, W^{\mathcal{E}})$ and $(\mathcal{F}, W^{\mathcal{F}})$. i.e. $\mathcal{G} = \mathcal{E} \otimes_C \mathcal{F}$. Then there are isometric product system morphism $I : \mathcal{E} \to \mathcal{G}$ and $J : \mathcal{F} \to \mathcal{G}$ such that the following holds:
(i) $\langle I_s(x), J_s(y) \rangle = \langle x, C_s y \rangle$ for all $x \in \mathcal{E}_s$ and $y \in \mathcal{F}_s$.
(ii) $\mathcal{G} = I(\mathcal{E}) \bigvee J(\mathcal{F})$.
Conversely, suppose $\mathcal{E}$ and $\mathcal{F}$ are two product subsystems of a product system $(\mathcal{H}, W)$. Then there is a contraction morphism $C : \mathcal{F} \to \mathcal{E}$ such that the amalgamated product $\mathcal{G}$ of $\mathcal{E}$ and $\mathcal{F}$ via C is isomorphic via ϕ to the product system generated by $\mathcal{E}$ and $\mathcal{F}$. i.e. $\mathcal{E} \otimes_C \mathcal{F} \sim \mathcal{E} \bigvee \mathcal{F}$ which is canonical in the sense that

$$\phi\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + b \, , a \in \mathcal{E}, b \in \mathcal{F}$$

Theorem

Let ϕ , ψ , τ be CP semigroups and (E, β) , (F, γ) , (G, δ) be their corresponding inclusion systems. Let $D_t : F_t \longrightarrow E_t$ by $D_t = P_{E_t}\pi_t(|a\rangle\langle b|)|_{F_t}$ (where P_{E_t} is the projection onto E_t). Then $D = \{D_t : t > 0\}$ is a contractive morphism from (F, γ) to (E, β). Moreover, (G, δ) is isomorphic to amalgamated sum of (E, β) and (F, γ) via D.

In this Section we will mainly concentrate on the category of product systems as defined by Arveson. In particular, all Hilbert spaces are separable and the product system has a measurable structure. By a contractive morphism, we mean a contractive morphism of product systems in that category i.e. the family of maps is a measurable family. We will call this category as category of Arveson's product systems.

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Define $\mathcal{C}(K_2, K_1) \subset \mathbb{C} \times K_2 \times K_1 \times B(K_2, K_1)$ as the set of all tuples (q, x, u, A) such that A is a contraction, $A^*u + x \in \text{Range}(I - A^*A)^{1/2}$, $q + \bar{q} \geq \|u\|^2 + q_0(A, x, u)$ where $q_0(A, x, u) = \inf\{\|a\|^2 : A^*x + u = (I - A^*A)^{1/2}a\}$. Equivalently,

$$\begin{pmatrix} q + \bar{q} - \|u\|^2 & -(A^*u + x)^* \\ -(A^*u + x) & I - A^*A \end{pmatrix} \geq 0$$

Parametrization of contractive morphism

Proposition

Suppose D is a contractive morphism from the product system $\Gamma(L^2[0,t], K_2)$ to the product system $\Gamma(L^2(0,t), K_1)$. Then there exists $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$ such that $D_t = [q, x, u, A]_t$. Conversely, for any tuple $(q, x, u, A) \in \mathcal{C}(K_2, K_1)$, $[q, x, u, A]_t$ defines a contractive morphism from $\Gamma(L^2(0,t), K_2)$ to $\Gamma(L^2(0,t), K_1)$.

Index computation

We will restrict ourselves into a subclass where the amalgamated product is an Arveson's product system. This is equivalent to the following assumption that there is a big Arveson's product system $\mathcal{H}$ which contains the two systems $\mathcal{E}$ and $\mathcal{F}$ as subsystems.

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Theorem

Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial Arveson product systems of index k_1 and k_2 respectively. Let $D : \mathcal{F} \to \mathcal{E}$ be a contractive morphism such that $\mathcal{E} \otimes_D \mathcal{F}$ is an Arveson product system. Then $D|_{\mathcal{F}^I} : \mathcal{F}^I \to \mathcal{E}^I$ is a contractive morphism. So they can be represented as $\mathcal{E}^I_t = \Gamma_{sym}(L^2[0,t], K_1)$ and $\mathcal{F}^I_t = \Gamma_{sym}(L^2[0,t], K_2)$. Then $D_t|_{\mathcal{F}^I_t} = [q,x,y,A]_t$ for some $(q,x,y,A) \in \mathcal{C}(K_2, K_1)$ and

$$ind\,(\mathcal{E} \otimes_D \mathcal{F}) = \begin{cases} \infty & \text{if } k_1 \text{ or } k_2 \text{ is } \infty \\ k_1 + k_2 - N(I - A^*A) & \text{if } q + \overline{q} - \|y\|^2 = \langle x + A^*y, a\rangle \\ & \text{where} (I - A^*A)a = x + A^*y \\ k_1 + k_2 - N(I - A^*A) + 1 & \text{otherwise} \end{cases}$$

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Powers' problem

Inclusion system

Amalgamation

contractive morphism

Index computation

Corollary

Suppose $\mathcal{E}$ and $\mathcal{F}$ are two spatial product systems of index k_1 and k_2 respectively. Let u^0 and v^0 be two units of $\mathcal{E}$ and $\mathcal{F}$ respectively such that $\|u_t^0\|, \|v_t^0\| \leq 1$ for all $t > 0$. Set $D_t = |u_t^0\rangle\langle v_t^0|$. Then $D_t : \mathcal{F}_t \to \mathcal{E}_t$ is a contractive morphism and

$$ind\,(\mathcal{E} \otimes_D \mathcal{F}) = \begin{cases} k_1 + k_2 & \text{if } \|u_t\| = \|v_t\| = 1 \text{ for all } t > 0 \\ k_1 + k_2 + 1 & \text{otherwise} \end{cases}$$

References

Bhat,Mukherjee,Inclusion systems and amalgamated product of product systems, IDAQP,March 2010.

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Thank you