Convolution Products of Linear Functionals

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Bangalore, August 2010

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1 Matrix exponentials

M_d : vector space of complex $d \times d$-matrices, $A = (a_{ij})_{i,j=1,\ldots,d} \in M_d$

Usual product of matrices

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$e^A = \lim_{n\to\infty} (E + \frac{A}{n})^n$$

Positivity

$A \geq 0$ means: $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, d\}$

$A, B \geq 0 \implies AB \geq 0$

Semigroups of matrices

$$T_t \in M_d : \; T_s T_t = T_{s+t}, \; \lim_{t\to 0} T_t = E \iff T_t = e^{tA} \text{ for some matrix } A \in M_d$$

Schoenberg corrrespondence

$$e^{tA} \geq 0 \text{ for all } t \in \mathbb{R}_+ \iff A \text{ is real and } a_{ij} \geq 0 \text{ for all } i \neq j$$

Using the dual pairing

$$(A, B) = \sum_{i,j} a_{ij}\, b_{ij}$$

we identify M_d and its dual vector space M_d'.

Turn M_d into a *-algebra by defining the product to be the *Schur product*, i.e. $A \cdot B$ is given by

$$(A \cdot B)_{ij} = a_{ij} b_{ij}$$

and the involution by $(A^*)_{ij} = \overline{a_{ij}}$.

Then $A \in M_d \cong M_d'$ is ≥ 0 iff $A \in M_d'$ is a positive linear functional on the above *-algebra M_d since

$$(A, B^* B) = \sum_{i,j} a_{ij} |b_{ij}|^2 \geq 0$$

for all $B \in M_d$ iff $a_{ij} \geq 0$ for all i, j.

Schur product of matrices

$$\mathrm{e}_{\cdot}^{A} := \sum_{n=1}^{\infty} \frac{A^{\cdot n}}{n!}$$

i.e. $(\mathrm{e}^A_\cdot)_{ij} = \mathrm{e}^{a_{ij}}$

$$\mathrm{e}^A_\cdot = \lim_{n\to\infty} (\mathbf{1} + \frac{A}{n})^{\cdot n}$$

Positivity

$A \geq 0$ means A is positive (semi-)definite.

$A, B \geq 0 \implies A \cdot B \geq 0$ (Result of Schur)

Semigroups of matrices

$$T_t \in M_d,\ T_s \cdot T_t = T_{s+t},\ \lim_{t\to 0} T_t = \mathbf{1} \iff T_t = \mathrm{e}^{tA}_\cdot \text{ for some matrix } A \in M_d$$

Schoenberg corrrespondence

$$\mathrm{e}^{tA}_\cdot \geq 0 \text{ for all } t \in \mathbb{R}_+ \iff A \text{ is hermitian and conditionally positive definite}$$

Using once more the dual pairing

$$(A, B) = \sum_{i,j} a_{ij}\, b_{ij}$$

we again identify M_d and its dual vector space M_d'.

Turn M_d into a *-algebra by defining the product to be the usual matrix multiplication and the involution by $(A^*)_{ij} = \overline{a_{ji}}$, i.e. A^* is the usual adjoint of the matrix A.

Then $A \in M_d \cong M_d'$ is ≥ 0 iff $A \in M_d'$ is a positive linear functional on the 'usual' *-algebra M_d since

$$(A, B^*B) = \sum_{i,j,n} a_{ij}\overline{b_{ni}}\,b_{nj} \geq 0$$

for all $B \in M_d$ iff A is positive definite.

2 Exponentials of linear functionals on coalgebras and bialgebras

Fundamental facts on coalgebras
(follow from the Fundamental Theorem on Coalgebras)

Let $(\mathcal{C}, \Delta, \delta)$ be a coalgebra with comultiplication Δ and counit δ.
The convolution product of two linear functionals φ_1 and φ_2 on $\mathcal{C}$ is defined by

$$\varphi_1 \star \varphi_2 = (\varphi_1 \otimes \varphi_2) \circ \Delta$$

Then, with respect to convolution, the space $\mathcal{C}'$ of linear functionals on $\mathcal{C}$ becomes a unital algebra.

1) The series

$$\sum_{n=0}^{\infty} \frac{\psi^{\star n}(c)}{n!} =: \mathrm{e}_{\star}^{\psi}(c)$$

converges for each $\psi \in \mathcal{C}'$ and $c \in \mathcal{C}$.

2) Let φ_t be a weakly continuous convolution semigroup of linear functionals on $\mathcal{C}$, i.e.

$$\varphi_s \star \varphi_t = \varphi_{s+t} \quad \text{and} \quad \lim_{t \to 0+} \varphi(c) = \delta(c) \text{ for all } \; c \in \mathcal{C}$$

Then

$$\psi(c) = \frac{\mathrm{d}}{\mathrm{d}t}\, \varphi_t(c)|_{t=0}$$

exists and

$$\varphi_t(c) = \mathrm{e}_\star^{t\psi}(c)$$

3) We have

$$\lim_{n\to\infty} (\delta + \frac{\psi}{n})^{\star n} = \mathrm{e}_\star^{\psi} \;\; \textit{weakly} \text{ (which means pointwise)}$$

and more:

$$\lim_{n\to\infty} (\delta + \frac{\psi}{n} + O(n^{-2}))^{\star n} = \mathrm{e}_\star^{\psi} \;\; \textit{weakly}$$

when $O(n^{-2})$ is a linear functional on $\mathcal{C}$ such that for $c \in \mathcal{C}$

$$|O(n^{-2})(c)| \leq D_c\, n^{-2}$$

for some constant D_c (depending on c).

Positivity

Let $(\mathcal{B}, \Delta, \delta)$ be a *-bialgebra, which means that $\mathcal{B}$ is a coalgebra and a *-algebra such that Δ and δ are *-algebra homomorphisms.

A linear functional φ on $\mathcal{B}$ is called positive if $\varphi(b^*b) \geq 0$ for all $b \in \mathcal{B}$, and we have

$$\varphi_1, \varphi_2 \text{ positive} \implies \varphi_1 \star \varphi_2 \text{ positive}$$

A linear functional ψ on $\mathcal{B}$ is called conditionally positive if $\psi(b^*b) \geq 0$ for all $b \in \operatorname{kern} \delta$

Schoenberg correspondence

$$\mathrm{e}_\star^{t\psi} \text{ are positive for all } t \in \mathbb{R}_+ \iff \psi \text{ is hermitian and conditionally positive}$$

Two different *-bialgebra structures on M_d give the Schoenberg correspondence results for the introductary examples of matrix and Schur multiplication:

Usual product of matrices

$$\Delta e_{ij} = \sum_{n=1}^{d} e_{in} \otimes e_{nj} \quad \text{and} \quad \delta e_{ij} = \delta_{ij}$$

Schur product of matrices

$$\Delta e_{ij} = e_{ij} \otimes e_{ij} \quad \text{and} \quad \delta e_{ij} = 1$$

3 Polynomial products

I : some index set
$\mathbb{C}^I = \{(\alpha_k)_{k\in I} \,|\, \alpha_k \in \mathbb{C}\}$
$\mathbb{C}[x_k, y_k]$: polynomial algebra in (commuting) indeterminates x_k and y_k, $k \in I$

Polynomial monoid structures

$P_l = P_l(x_k, y_k)$, $l \in I$: polynomials in x_k, y_k satisfying

$$P_l(P_k(x_m, y_m), z_k) = P_l(x_k, P_k(y_m, z_m)) \tag{1}$$

and ($0 \in \mathbb{C}^I$ denotes the family consisting of 0's)

$$P_l(x_k, 0) = x_l \;\text{ and }\; P_l(0, y_k) = y_k \tag{2}$$

We put, for $\alpha, \beta \in \mathbb{C}^I$,

$$(\alpha \star \beta)_l := P_l(\alpha_k, \beta_k)$$

to define a 'convolution' product which as a consequence of properties (1) and (2) of the polynomials turns $\mathbb{C}^I$ into a monoid with unit element 0, i.e.

$$(\alpha \star \beta) \star \gamma = \alpha \star (\beta \star \gamma)$$

and

$$\alpha \star 0 = \alpha = 0 \star \alpha$$

For $\alpha \in \mathbb{C}^I$ and $n \in \mathbb{N}$ denote by α/n the element of $\mathbb{C}^I$ with $(\alpha/n)_k = \frac{\alpha_k}{n}$.

Theorem
Let $\alpha \in \mathbb{C}^I$ and $k \in I$.
Then the sequence $((\alpha/n)^{\star n})_k$ converges if $n \to \infty$.

Proof : Define the algebra homomorphism

$$\Delta : \mathbb{C}[x_k] \to \mathbb{C}[x_k] \otimes \mathbb{C}[x_k] \cong \mathbb{C}[x_k, y_k]$$

by

$$\Delta x_k = P_k$$

Then (1) and (2) imply that $(\mathbb{C}[x_k], \Delta, S(0))$ forms a (commutative) bialgebra where $S(0)$ is defined by $S(0)P(x_k) = P(0)$.
More generally, for $\alpha \in \mathbb{C}^I$, we define $S(\alpha) \in \mathbb{C}[x_k]'$ by $S(\alpha)P(x_k) = P(\alpha_k)$.
Then

$$\begin{aligned} S(\alpha \star \beta)(x_l) &= (\alpha \star \beta)_l \\ &= P_l(\alpha_k, \beta_k) \\ &= (S(\alpha) \otimes S(\beta))(\Delta x_l) \\ &= (S(\alpha) \star S(\beta))(x_l) \end{aligned}$$

and

$$((\alpha/n)^{\star n})_k = S(\alpha/n)^{\star n}(x_k)$$

Claim:

$$S(\alpha/n) = S(0) + \frac{D(\alpha)}{n} + O(n^{-2})$$

where for a monomial $M \in \mathbb{C}[x_k]$ we put

$$D(\alpha)(M) = \begin{cases} \alpha_k & \text{if } M = x_k \\ 0 & \text{otherwise} \end{cases}$$

Proof of claim: We evaluate $S(\alpha/n)$ at a monomial M.

$$\begin{aligned} M = \mathbf{1} &\ :\ S(\alpha/n)(\mathbf{1}) = 1 = S(0)(\mathbf{1}) \\ M = x_l &\ :\ S(\alpha/n)(x_l) = (\alpha/n)_l = D(\alpha/n)(x_l) = \frac{D(\alpha)}{n}(x_l) \\ \deg M \geq 2 &\ :\ S(\alpha/n)(M) = \frac{1}{n^{\deg M}} S(\alpha)(M) = O(n^{-2})(M) \end{aligned}$$

Now

$$S(\alpha/n)^{\star n} \longrightarrow \mathrm{e}_\star^{D(\alpha)} \quad \textit{weakly}$$

follows and, in particular,

$$((\alpha/n)^{\star n})_k \longrightarrow \mathrm{e}_\star^{D(\alpha)}(x_k) \ \square$$

It follows from property (1) that for fixed $\alpha_1, \dots, \alpha_n \in \mathbb{C}^I$, $k \in I$, the function

$$(t_1, \dots, t_n) \mapsto ((t_1\alpha_1) \star \cdots \star (t_n\alpha_n))_k$$

is a polynomial in the real quantities $t_1, \dots, t_n$.
Thus we may define the n^{th} $\boxplus$-product by

$$\begin{aligned}(\alpha_1 \boxplus \cdots \boxplus \alpha_n)_k : &= \frac{\partial^n}{\partial t_1 \cdots \partial t_n} ((t_1\alpha_1) \star \cdots \star (t_n\alpha_n))_k \big|_{t_1=\cdots=t_n=0} \\ &= (D(\alpha_1) \star \cdots \star D(\alpha_n))(x_k)\end{aligned}$$

to write

$$(\mathrm{e}_\star^\alpha)_k := \mathrm{e}_\star^{D(\alpha)}(x_k)$$

and

$$\mathrm{e}_\star^\alpha = \sum_{n=0}^\infty \frac{\alpha^{\boxplus n}}{n!}$$

We have

$$S(\mathrm{e}_\star^\alpha) = \mathrm{e}_\star^{D(\alpha)}$$

Theorem

$$\alpha \boxplus \beta = \beta \boxplus \alpha \iff \mathrm{e}_\star^{\alpha+\beta} = \mathrm{e}_\star^{\alpha} \star \mathrm{e}_\star^{\beta}$$

Proof: '$\Rightarrow$' : For a coalgebra $\mathcal{C}$ call a linear functional d on $\mathcal{C}$ a derivation if

$$d(bc) = d(b)\delta(c) + \delta(b)d(c)$$

for all $b, c \in \mathcal{C}$.
Then

$$d_1, d_2 \text{ derivations} \implies [d_1, d_2]_\star \ (= d_1 \star d_2 - d_2 \star d_1) \text{ is a derivation}$$

Suppose now that $[\alpha, \beta]_{\boxplus} = 0$.
Then

$$0 = ([\alpha, \beta])_k = [D(\alpha), D(\beta)]_\star(x_k) \implies [D(\alpha), D(\beta)]_\star = 0$$

since a derivation is determined by its values on the generators x_k.
From $[D(\alpha), D(\beta)]_\star = 0$ we obtain

$$\begin{aligned} \mathrm{e}_\star^{D(\alpha+\beta)} &= \mathrm{e}_\star^{D(\alpha)+D(\beta)} \\ &= \mathrm{e}_\star^{D(\alpha)} \star \mathrm{e}_\star^{D(\beta)} \end{aligned}$$

and

$$\begin{aligned} S(\mathrm{e}_\star^\alpha \star \mathrm{e}_\star^\beta) &= S(\mathrm{e}_\star^\alpha) \star S(\mathrm{e}_\star^\beta) \\ &= \mathrm{e}_\star^{D(\alpha)} \star \mathrm{e}_\star^{D(\beta)} \\ &= \mathrm{e}_\star^{D(\alpha+\beta)} \\ &= S(\mathrm{e}_\star^{\alpha+\beta}) \ \square \end{aligned}$$

It follows that for $\beta \in \mathbb{C}^I$ the functionals $\alpha_t = \mathrm{e}_\star^{t\beta}$, $t \in \mathbb{R}_+$, form a 1-parameter semigroup which is weakly continuous, i.e. $\lim_{t\to 0}(\alpha_t)_k = 0$ for all $k \in I$.
Conversely, by passing to the weakly continuous convolution semigroup $S(\alpha_t)$ on the coalgebra $\mathbb{C}[x_k]$ we have

Theorem
Let α_t be a weakly continuous 1-parameter semigroup in $\mathbb{C}^I$.
Then

$$\beta_k = \frac{\mathrm{d}}{\mathrm{d}t}\,(\alpha_t)_k|_{t=0}$$

exists for all $k \in I$ and

$$\alpha_t = \mathrm{e}_\star^{t\beta}$$

4 Exponentials of linear functionals on Dual Semigroups

We give a realization of the category coproduct of two algebras $\mathcal{B}_1$, $\mathcal{B}_2$ which is the free product of the algebras or, in other words, the algebra freely generated by the algebras $\mathcal{B}_1$ and $\mathcal{B}_2$.
As a vector space

$$\mathcal{B}_1 \sqcup \mathcal{B}_2 = \bigoplus_{\varepsilon \in \mathbb{A}} \mathcal{B}_\varepsilon$$

where

$$\mathbb{A} = \{(\varepsilon_1, \ldots, \varepsilon_m) \,|\, m \in \mathbb{N}; \varepsilon_i = 1, 2; \varepsilon_i \neq \varepsilon_{i+1}\}$$

and, for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$,

$$\mathcal{B}_\varepsilon = \mathcal{B}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{B}_{\varepsilon_m}.$$

The multiplication is given by

$$
\begin{aligned}
&(a_1 \otimes \cdots \otimes a_m)\,(b_1 \otimes \cdots \otimes b_n) \\
&\quad = \begin{cases} (a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n) & \text{if } \varepsilon_m \neq \gamma_1 \\ (a_1 \otimes \cdots \otimes a_{m-1} \otimes (a_m b_1) \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_n) & \text{if } \varepsilon_m = \gamma_1 \end{cases}
\end{aligned}
$$

for

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m),\ \gamma = (\gamma_1, \ldots, \gamma_n)$$

and

$$a_1 \otimes \cdots \otimes a_m \in \mathcal{B}_\varepsilon,\ b_1 \otimes \cdots \otimes b_n \in \mathcal{B}_\gamma.$$

Notice that, like for the tensor case, in a very natural way,

$$(\mathcal{B}_1 \sqcup \mathcal{B}_2) \sqcup \mathcal{B}_3 \cong \mathcal{B}_1 \sqcup (\mathcal{B}_2) \sqcup \mathcal{B}_3)$$

and that, even in a more natural way than for the tensor case,

$$\mathcal{B}_1 \sqcup \mathcal{B}_2 \cong \mathcal{B}_2 \sqcup \mathcal{B}_1.$$

Noncommutative notions of independence

In this talk such a notion is given by a 'universal' product, i.e. a prescription which associates with each pair $\varphi_1 : \mathcal{B}_1 \to \mathbb{C}, \varphi_2 : \mathcal{B}_2 \to \mathbb{C}$ of linear functionals $\varphi_{1/2}$ on algebras $\mathcal{B}_{1/2}$ a linear functional $\varphi_1 \bullet \varphi_2$ on the coproduct $\mathcal{B}_1 \sqcup \mathcal{B}_2$ such that

$$
\begin{aligned}
(\varphi_1 \bullet \varphi_2) \circ \iota_1 &= \varphi_1 && (3) \\
(\varphi_1 \bullet \varphi_2) \circ \iota_2 &= \varphi_2 && \\
(\varphi_1 \bullet \varphi_2) \bullet \varphi_3 &= \varphi_1 \bullet (\varphi_2 \bullet \varphi_3) && (4) \\
(\varphi_1 \circ j_1) \bullet (\varphi_2 \circ j_2) &= (\varphi_1 \bullet \varphi_2) \circ (j_1 \sqcup j_2) && (5)
\end{aligned}
$$

If $(\mathcal{A}, \Phi)$ is a quantum probability space, the *-subalgebras $\mathcal{A}_{1/2}$ of $\mathcal{A}$ are called independent if the joint distribution $\Phi \circ (\iota_1 \sqcup \iota_2)$ of the embeddings $\iota_{1/2}$ is given by the product $(\Phi \circ \iota_1) \bullet (\Phi \circ \iota_2)$ of the marginal distributions.

Voiculescu Dual Semigroups and Groups

A Dual Semigroup is a *-algebra $\mathcal{B}$ together with a *-algebra homomorphism

$$\Delta \ : \ \mathcal{B} \to \mathcal{B} \sqcup \mathcal{B}$$

such that

$$\begin{aligned} (\Delta \sqcup \mathrm{id}) \circ \Delta &= (\mathrm{id} \sqcup \Delta) \circ \Delta \\ (0 \sqcup \mathrm{id}) \circ \Delta &= \mathrm{id} = (\mathrm{id} \sqcup 0) \circ \Delta. \end{aligned}$$

If an antipode exists, i.e. a *-algebra homomorphism $\mathrm{A} : \mathcal{B} \to \mathcal{B}$ such that

$$M \circ (\mathrm{A} \sqcup \mathrm{id}) \circ \Delta = 0 = M \circ (\mathrm{id} \sqcup \mathrm{A}) \circ \Delta,$$

the Dual Semigroup $\mathcal{B}$ is called a Dual Group.

Symmetric tensor algebras

Let V be a vector space and denote by $S(V)$ the symmetric tensor algbera over V.

This object can be characterized by the following universal property.
There is an embedding ι of V into $S(V)$ such for each linear mapping

$$R : V \to \mathcal{D}$$

from V to a commutative unital algebra $\mathcal{D}$ there exists a unique unital algebra homomorphism $S(R)$ with $S(R) \circ \iota = R$.

$S(V)$ can be realized as follows.
Choose a vector space basis $\{v_k \mid k \in I\}$ and put $S(V) = \mathbb{C}[x_k]$.
Then the embedding is the obvious one, and $S(R)$ is the homomorphism with

$$S(R)(x_k) = R(v_k).$$

Given I we may put $V = \mathbb{C}I$.
Then $V' = \mathbb{C}^I$, and a polynomial monoid structure on $\mathbb{C}^I = V'$ is nothing but a bialgebra structure on $S(V)$ with counit $S(0)$.

It can be shown that an equivalent formulation of universal products is by families

$$\sigma_{\mathcal{B}_1,\mathcal{B}_2} : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to S(\mathcal{B}_1) \otimes S(\mathcal{B}_2),$$

the relation to the universal product being given by the equation

$$\varphi_1 \bullet \varphi_2 = (S(\varphi_1) \otimes S(\varphi_2)) \circ \sigma_{\mathcal{B},\mathcal{B}}$$

Given a Dual Semigroup $\mathcal{B}$ and a notion of noncommutative independence we define

$$\varphi_1 \star \varphi_2 := (\varphi_1 \bullet \varphi_2) \circ \Delta$$

to obtain a polynomial convolution product for linear functionals on the Dual Semigroup $\mathcal{B}$ (**see the talks of Stephanie Lachs and Stefan Voß !**).

Example

$\mathcal{B} = \mathbb{C}_0[x] = \{P \in \mathbb{C}[x] \mid P(0) = 0\}$
$\Delta x = \iota_1(x) + \iota_2(x)$; $\iota_{1/2} : \mathcal{B} \to \mathcal{B} \sqcup \mathcal{B}$ the canonical embeddings
We choose Boolean independence, so that

$$(\varphi_1 \bullet \varphi_2)(b_1 \otimes \cdots \otimes b_m) = \varphi_{\varepsilon_1}(b_1) \cdots \varphi_{\varepsilon_m}(b_m)$$

for $b_1 \otimes \cdots \otimes b_m \in \mathcal{B}_\varepsilon$, $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$.
Then

$$\begin{aligned}(\varphi_1 \star \varphi_2)(x^n) &= (\varphi_1 \bullet \varphi_2)(\iota_1(x) + \iota_2(x))^n \\ &= \sum_{k=1}^{n} \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N} \\ l_1+\cdots+l_k=n}} \big(\varphi_1(x^{l_1})\varphi_2(x^{l_2})\varphi_1(x^{l_3}) \cdots + \varphi_2(x^{l_1})\varphi_1(x^{l_2})\varphi_2(x^{l_3}) \cdots \big)\end{aligned}$$

which gives

$$(\psi_1 \boxplus \cdots \boxplus \psi_k)(x^n) = k! \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N} \\ l_1+\cdots+l_k=n}} \psi_1(x^{l_1}) \cdots \psi_k(x^{l_k})$$

Finally, we obtain

$$\psi^{\boxplus n} = n!\, \psi^{*n}$$

where $*$ denotes the coalgebra convolution coming from the coalgebra structure ('shuffle algebra') on $\mathbb{C}[x]$ given by the comultiplication

$$x^n \mapsto \sum_{k=1}^{n} x^k \otimes x^{n-k}$$

so that in this example

$$\mathrm{e}_{\star}^{\psi} = \sum_{n=0}^{\infty} \psi^{*n}$$

Schoenberg correspondence

Let there be given

- a positive (!) universal product (satsifying (3)-(5))
- a Dual Semigroup $\mathcal{B}$

Theorem

For a linear functional ψ on $\mathcal{B}$ we have

$$\mathrm{e}_\star^{t\psi} \textit{ are positive for all } t \in \mathbb{R}_+ \iff \psi \textit{ is hermitian and conditionally positive}$$

Outline of proof : By Muraki's classification theorem there are exactly five positive universal products.
Schoenberg correspondence holds in all five cases for Dual Groups of tensor algebra type. (Use the realization of the GNS representation on a suitable Fock space.)

Now use the transformation theory for quantum Lévy processes (**see talk of Stefan Voß !**) to prove the Schoenberg correspondence in the general case.

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