Additive Deformations of Bialgebras

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Content

Bialgebras

Def. (bialgebra)

- unital algebra $(\mathcal{B}, \mu, \mathbb{1})$
- algebra homomorphism $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ (comultiplication) s.t.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

- algebra homomorphism $\delta : \mathcal{B} \to \mathbb{C}$ (counit) s.t.

$$(\delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \delta) \circ \Delta = \mathrm{id}$$

Def. (convolution)

- for $R, M : \mathcal{B}^{\otimes n} \to \mathcal{B}^{\otimes m}$ define $R \star M := \mu_m \circ (R \otimes M) \circ \Delta_n$
- for $\varphi, \psi : \mathcal{B} \to \mathbb{C}$ this simplifies to $\varphi \star \psi := (\varphi \otimes \psi) \circ \Delta$

More on Convolution

Def. (pointwise continuous convolution semigroup)

- $(\varphi_t)_{t \in \mathbb{R}}$
- $\varphi_t(a) \xrightarrow{t \to 0} \delta(a) \quad \forall a \in \mathcal{B}$
- $\varphi_t \star \varphi_s = \varphi_{t+s} \quad \forall t, s$

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Then:

- $\psi := \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big|_{t=0}$ exists pointwise
- $\varphi_t = \exp_\star(t\psi) = \delta + t\psi + \frac{t^2}{2}\psi \star \psi + \dots$

as a consequence of the fundamental theorem for coalgebras

Heisenberg Algebra

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algebra $\mathcal{A}$ with generators $a, a^\dagger$ and commutation relation

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$\mathcal{A}$ bialgebra?

no bialgebra structure s.t.

$$\Delta a^{(\dagger)} = a^{(\dagger)} \otimes \mathbb{1} + \mathbb{1} \otimes a^{(\dagger)},$$

since

$$\left[\Delta a, \Delta a^\dagger\right] = \left[a, a^\dagger\right] \otimes \mathbb{1} + \mathbb{1} \otimes \left[a, a^\dagger\right] = 2\mathbb{1} \otimes \mathbb{1} \neq \Delta \left[a, a^\dagger\right]$$

A way out

Idea

define comultiplication as algebra homomorphism

$$\Delta : \mathcal{A}_{s+t} \to \mathcal{A}_s \otimes \mathcal{A}_t ,$$

- $\mathcal{A}_t$ is algebra with generators $a, a^\dagger$ and commutation relation

$$\left[a, a^\dagger\right]_t = t\mathbb{1}$$

- relations respected:

$$\left[a, a^\dagger\right]_s \otimes \mathbb{1} + \mathbb{1} \otimes \left[a, a^\dagger\right]_t = s\mathbb{1} \otimes \mathbb{1} + t\mathbb{1} \otimes \mathbb{1} = \Delta\left(\left[a, a^\dagger\right]_{t+s}\right)$$

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Note

For $*$-bialgebras $(\mathcal{B}, \mu_t, \mathbb{1}, *)$ should be $*$-algebras and Δ should be a $*$-homomorphism.

Note

- the last condition can be written as

$$\Delta \circ \mu_{t+s} = \underbrace{(\mu_t \otimes \mu_s) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})}_{\text{multiplication on } \mathcal{B}_t \otimes \mathcal{B}_s} \circ (\Delta \otimes \Delta)$$

- applying the counit $\delta \otimes \delta$ to the last condition yields

$$\begin{aligned} \delta \circ \mu_{t+s} &= ((\delta \circ \mu_t) \otimes (\delta \circ \mu_s)) \circ \underbrace{(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)}_{\text{comultiplication on } \mathcal{B} \otimes \mathcal{B}} \\ &= (\delta \circ \mu_t) \star (\delta \circ \mu_s) \end{aligned}$$

- $(\delta \circ \mu_t)_{t \geq 0}$ pointwise continuous convolution semigroup
  $\longrightarrow \delta \circ \mu_t = \exp_\star(tL)$

The Generator of an Additive Deformation

Thm. (J. Wirth, 2002)

1-1-correspondence between additive deformations and generators via the equations

$$L = \frac{\mathrm{d}}{\mathrm{d}t}\delta \circ \mu_t\bigg|_{t=0} \qquad \mu_t = \mu \star \exp_\star(tL)$$

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Last condition corresponds to associativity:
$\mu_t \circ (\mathrm{id} \otimes \mu_t) = \mu_t \circ (\mu_t \otimes \mathrm{id}) \quad \forall t$

Hopf Algebras

Def. (Hopf algebra)

- bialgebra $\mathcal{B}$
- linear map $S : \mathcal{B} \to \mathcal{B}$ s.t.

$$\underbrace{\mu \circ (S \otimes \mathrm{id}) \circ \Delta}_{S \star \mathrm{id}} = \underbrace{\mu \circ (\mathrm{id} \otimes S) \circ \Delta}_{\mathrm{id} \star S} = \mathbb{1}\delta$$

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question:

How does the antipode property carry over to additive deformations?

Deformed Antipodes

Def. (Hopf deformation)

exist linear mappings $S_t : \mathcal{B} \to \mathcal{B}$ s.t.

$$\mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \mathbb{1}\delta$$

Thm. G.

Every additive deformation of a Hopf algebra is a Hopf deformation.
Properties:

- $S_t(\mathbb{1}) = \mathbb{1}$
- $S_t : \mathcal{B}_{-t} \to \mathcal{B}_t$ is algebra antihomomorphism
- $(S_t \otimes S_r) \circ \tau \circ \Delta = \Delta \circ S_{t+r}$
- $(\delta \circ S_t)_{t \in \mathbb{R}}$ is pointwise continuous convolution semigroup

Idea of the proof

Assume we have a Hopf deformation and $\delta \circ S_t = \exp_\star(t\sigma)$.
$\longrightarrow$ apply δ to the deformed antipode condition:

$$\delta \circ \mu_t \circ (S_t \otimes \mathrm{id}) \circ \Delta = \delta \circ \mu_t \circ (\mathrm{id} \otimes S_t) \circ \Delta = \delta$$

$\longrightarrow$ differentiate at $t = 0$:

$$L \circ (S \otimes \mathrm{id}) + (\sigma \otimes \delta) \circ \Delta = L \circ (\mathrm{id} \otimes S) + (\delta \otimes \sigma) \circ \Delta = 0$$

or equivalently

$$\sigma = -L \circ (S \otimes \mathrm{id}) = -L \circ (\mathrm{id} \otimes S).$$

$\longrightarrow$ We have to prove that $S \star \exp_\star(-tL \circ (S \otimes \mathrm{id}))$ fulfills the condition for the deformed antipodes.
The extra conditions for the S_t are proven similarly as for the usual antipode.

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For an additive deformation $\mu_t = \mu \star \exp_\star(tL)$

$$[x_1, x_2]_t = \mu \star \exp_\star(tL)(x_1 \otimes x_2 - x_2 \otimes x_1) = tL(x_1 \otimes x_2 - x_2 \otimes x_1)\mathbb{1}$$

Proposition

$L(x_1 \otimes x_2) = 1/2$, $L(x_2 \otimes x_1) = -1/2$ and $L(M \otimes N) = 0$ on all other monomials defines a generator of an additive deformation s.t. $\mathcal{B}_t \cong \mathcal{A}_t$

q-Heisenberg algebra

Given:

- algebra with generators x, y, h, h^{-1} with relations

$$xy = qyx \qquad hh^{-1} = \mathbb{1} = h^{-1}h$$
$$xh = qhx \qquad yh = qhy$$

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$$\Delta(y) = y \otimes \mathbb{1} + h \otimes y \qquad \delta(y) = 0 \qquad S(y) = -y$$
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Proposition

$L(h^k x \otimes yh^l) = 1/2$, $L(h^k y \otimes xh^l) = -1/(2q)$ and $L(M \otimes N) = 0$ for all other monomials defines a generator with $\mu_t(x \otimes y - qy \otimes x) = t\mathbb{1}$

A Short Bibliography

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